

PFAFFIAN SUM FORMULA FOR THE SYMPLECTIC GRASSMANNIANS

TAKESHI IKEDA AND TOMOO MATSUMURA

ABSTRACT. We study the torus equivariant Schubert classes of the Grassmannian of non-maximal isotropic subspaces in a symplectic vector space. We prove a formula that expresses each of those classes as a *sum* of multi Schur-Pfaffians, whose entries are equivariantly modified special Schubert classes. Our result gives a proof to Wilson’s conjectural formula, which generalizes the Giambelli formula for the ordinary cohomology proved by Buch-Kresch-Tamvakis, given in terms of Young’s raising operators. Furthermore we show that the formula extends to a certain family of Schubert classes of the symplectic partial isotropic flag varieties. Schubert classes and Symplectic Grassmannians and Torus equivariant cohomology and Giambelli type formula and Wilson’s conjecture and Double Schubert polynomials

1. INTRODUCTION

The classical Giambelli formula [8] expresses a general Schubert class of the Grassmannian as the determinant of a matrix whose entries are the so-called *special Schubert classes*. A special Schubert class is defined by the locus of subspaces having excess intersection with a fixed subspace. These classes also coincide with the Chern classes of the universal quotient bundle over the Grassmannian. Various extensions of the formula have been obtained (see for example [7], [23] and the references therein). The *Giambelli problem* consists in finding a “closed formula” for a Schubert class in terms of those special classes. Note that, in the torus equivariant setting, the problem is closely related to the theory of degeneracy loci of vector bundles (*cf.* [1], [7], [23]).

For the symplectic or orthogonal Grassmannians, there is a natural notion of special Schubert classes, which takes into account the isotropic condition. For the Grassmannian of maximal isotropic subspaces, the Giambelli formula, first found by Pragacz [20], expresses a general Schubert class as a Pfaffian whose entries are appropriate quadratic polynomials in the special Schubert classes. Its natural equivariant version was obtained by Kazarian [15] in the context of degeneracy loci (see [16], [21] for other versions, and also the survey articles [7], [23] and references therein). The Kazarian’s formula was rediscovered by Ikeda in [10] and Ikeda-Naruse in [12] and was interpreted in the context of the torus equivariant cohomology. For the non-maximal isotropic Grassmannians, an answer to the (non-equivariant) Giambelli problem was given by Buch, Kresch, and Tamvakis ([4], [5]). Their formula expresses an arbitrary Schubert class by means of *Young’s raising operators*. We can regard this polynomial expression as a certain “combinatorial interpolation” between the Jacobi-Trudi determinant and the Schur Pfaffian.

This paper is concerned with the *equivariant* Giambelli problem for the non-maximal isotropic Grassmannians in the symplectic case. In [26], Wilson employed the raising operators to define *double theta polynomials* Θ_λ , and proved that these polynomials satisfy the equivariant Chevalley formula for the non-maximal symplectic Grassmannian. In [26], it was further conjectured that Θ_λ would equal to the *double Schubert polynomial* introduced in [10] (see §1.2).

The main result of this paper provides a formula expressing each double Schubert polynomial associated to the isotropic Grassmannians as a sum of Pfaffians whose entries are Wilson's double theta polynomials corresponding to the special Schubert classes. This immediately leads to a proof of Wilson's conjecture, because the raising operator formula can be rewritten as a Pfaffian sum by a formal computation (see §7 for details). Note that the equivalence of the two formulas in the non-equivariant situation was already included in the proof of Proposition 2 of [6]. In this sense, the non-equivariant version of the Pfaffian sum formula for the symplectic Grassmannian was first obtained in [6].

Our method for the proof of the main result is to use the *left divided difference operators*. These operators are essential in the theory of (double) Schubert polynomials, and exist only in the equivariant setup. This technique allows us to completely avoid using the raising operators, and more importantly, the technique is applicable to more general contexts. In particular, we extend the Pfaffian sum formula beyond Grassmannians, namely to some part of the Schubert classes of the partial isotropic flag varieties. In this extended Pfaffian sum formula, each entry can be regarded as equivariantly modified special Schubert classes arising from the symplectic Grassmannians of various dimensional subspaces.

Below we summarize our results in more details.

1.1. Symplectic Grassmannian and its Schubert varieties. Throughout the paper, we fix a non-negative integer k . For any positive integer $n \geq k$, let SG_n^k denote the Grassmannian of $(n - k)$ -dimensional isotropic subspaces in \mathbb{C}^{2n} equipped with a symplectic form. There is a maximal parabolic subgroup P_k of the symplectic group $G := Sp_{2n}(\mathbb{C})$ such that SG_n^k can be realized as the homogeneous space G/P_k .

A partition λ is k -strict if no part greater than k is repeated. The Schubert varieties of SG_n^k are indexed by the k -strict partitions whose Young diagrams fit in the $(n - k) \times (n + k)$ rectangle. We denote the set of such partitions by $\mathcal{P}_n^{(k)}$. Given $\lambda \in \mathcal{P}_n^{(k)}$ and a complete flag of subspaces $0 = F_0 \subset F_1 \subset \cdots \subset F_{2n} = \mathbb{C}^{2n}$ such that $F_{n+i} = (F_{n-i})^\perp$ for $0 \leq i \leq n$, the corresponding Schubert variety is defined as

$$\Omega_\lambda = \{V \in SG_n^k \mid \dim(V \cap F_{p_j(\lambda)}) \geq j, \quad 1 \leq \forall j \leq \ell(\lambda)\}, \quad (1.1)$$

where $\ell(\lambda)$ denotes the number of non-zero parts of λ , and

$$p_j(\lambda) = n + k + j - \lambda_j - \#\{i \mid i < j, \lambda_i + \lambda_j > 2k + j - i\}.$$

See [4], for example. The codimension of Ω_λ is $|\lambda| = \sum_i \lambda_i$. The corresponding class $[\Omega_\lambda] \in H^*(SG_n^k)$ is the Schubert class. The *special Schubert varieties* are the ones associated to one-line k -strict partitions:

$$\Omega_r = \{V \in SG_n^k \mid \dim(V \cap F_{n+k+1-r}) \geq 1\},$$

for $1 \leq r \leq n + k$. Their classes $[\Omega_r]$ are called the *special Schubert classes*. They are equal to the r -th Chern classes $c_r(\mathcal{Q})$ of the universal quotient bundle \mathcal{Q} over SG_n^k .

1.2. Double Schubert polynomials of type C. We recall the results in [11], where the double Schubert polynomials of type C were introduced. Let B be a Borel subgroup of $G = Sp_{2n}(\mathbb{C})$, and T the maximal torus contained in B . The Weyl group $N_G(T)/T$ is denoted by W_n and identified with the group of the signed permutations of $\{\pm 1, \dots, \pm n\}$. We often denote $-i$ by \bar{i} . The flag variety $\mathcal{F}l_n$ is defined as the quotient space G/B . For each $w \in W_n$, the Schubert variety X_w is

defined as the Zariski closure of B^- -orbit of the corresponding point $e_w \in \mathcal{F}l_n$, where B^- is the Borel subgroup such that $B \cap B^- = T$. The codimension of X_w is precisely the length $\ell(w)$ of w as a Weyl group element of W_n . We denote by $[X_w]_T$ the corresponding T -equivariant Schubert class in $H_T^*(\mathcal{F}l_n)$.

Let Γ be the ring generated over \mathbb{Z} by the Schur Q -functions $Q_r(x)$ ($r \geq 1$), where x is an infinite sequence of variables x_1, x_2, \dots . Let \mathcal{R}_∞ be the polynomial ring $\Gamma[z, t]$ in the variables z_i, t_i ($i \geq 1$) with coefficients in Γ . Let W_∞ be the Weyl group of type C_∞ , where we can regard it as the union of W_n ($n \geq 1$). There are two commuting actions of W_∞ on the ring \mathcal{R}_∞ (see §2). The *double Schubert polynomials* are defined as elements of a distinguished $\mathbb{Z}[t]$ -basis $\{\mathfrak{C}_w(z, t; x) \mid w \in W_\infty\}$ of \mathcal{R}_∞ . They are characterized by two series of operators $\{\delta_i \mid i \geq 0\}$ and $\{\partial_i \mid i \geq 0\}$ called the left and the right divided difference operators respectively (See Theorem 2.1 in Section 2).

The integral T -equivariant cohomology ring $H_T^*(\mathcal{F}l_n)$ of $\mathcal{F}l_n$ has an $H_T^*(pt)$ -algebra structure given by the pullback of $\mathcal{F}l_n \rightarrow pt$. Together with an appropriate identification $H_T^*(pt) = \mathbb{Z}[t_1, \dots, t_n]$, there is a canonical homomorphism

$$\pi_n : \mathcal{R}_\infty \longrightarrow H_T^*(\mathcal{F}l_n)$$

of $H_T^*(pt)$ -algebras such that π_n sends $\mathfrak{C}_w(z, t; x)$ to $[X_w]_T$ if $w \in W_n$ and to zero if $w \notin W_n$.

1.3. Equivariant Schubert classes of $H_T^*(SG_n^k)$. Let s_i ($i \geq 0$) be the standard (Coxeter) generators of W_∞ , usually referred to as the simple reflections (see §2). Let $W_{(k)}$ be the subgroup of W_∞ generated by s_i ($i \geq 0, i \neq k$). Let $\mathcal{R}_\infty^{(k)}$ denote the invariant subring of \mathcal{R}_∞ with respect to the 2nd (“right”) action of $W_{(k)}$. There is the following commutative diagram

$$\begin{array}{ccc} \mathcal{R}_\infty^{(k)} & \xrightarrow{\quad} & \mathcal{R}_\infty \\ \downarrow \pi_n^{(k)} & & \downarrow \pi_n \\ H_T^*(SG_n^k) & \xrightarrow{pr_k^*} & H_T^*(\mathcal{F}l_n), \end{array}$$

where the horizontal arrow pr_k^* in the second row is the pullback of the natural projection $pr_k : \mathcal{F}l_n \rightarrow SG_n^k$, and $\pi_n^{(k)}$ is obtained by restricting π_n to $\mathcal{R}_\infty^{(k)}$.

Let $W_\infty^{(k)}$ be the set of the minimum-length coset representatives for $W_\infty/W_{(k)}$. We denote the set of all k -strict partitions by $\mathcal{P}_\infty^{(k)} = \bigcup_{n \geq k} \mathcal{P}_n^{(k)}$. There is a natural bijection

$$\mathcal{P}_\infty^{(k)} \longrightarrow W_\infty^{(k)}, \quad \lambda \mapsto w_\lambda^{(k)},$$

such that $|\lambda| = \ell(w_\lambda^{(k)})$ and the image of $\mathcal{P}_n^{(k)}$ is $W_n^{(k)} := W_n \cap W_\infty^{(k)}$. If $\lambda \in \mathcal{P}_n^{(k)}$, we have $pr_k^*[\Omega_\lambda]_T = [X_{w_\lambda^{(k)}}]_T$ and

$$\pi_n^{(k)}(\mathfrak{C}_{w_\lambda^{(k)}}(z, t; x)) = [\Omega_\lambda]_T.$$

In particular, the special Schubert class $[\Omega_r]_T$ of degree r is the image of $\mathfrak{C}_{w_r^{(k)}}(z, t; x)$, where $w_r^{(k)}$ is the element of $W_\infty^{(k)}$ corresponding to the partition with r boxes in one row. The set of the functions $\mathfrak{C}_w(z, t; x)$, $w \in W_\infty^{(k)}$ forms a $\mathbb{Z}[t]$ -basis of $\mathcal{R}_\infty^{(k)}$ (Proposition 3.7).

1.4. Main results. Our goal is to give an explicit closed formula describing $\mathfrak{C}_w(z, t; x)$ ($w \in W_\infty^{(k)}$) as a polynomial in terms of the double Schubert polynomials corresponding to the special classes $[\Omega_r]_T$ ($r \geq 1$).

Definition 1.1. Define ${}_k\vartheta_r^{(l)}(x, z|t)$ for $l, r \geq 0$ by

$$\begin{aligned} \sum_{r=0}^{\infty} {}_k\vartheta_r^{(l)}(x, z|t) \cdot u^r &= \prod_{i=1}^{\infty} \frac{1+x_i u}{1-x_i u} \prod_{i=1}^k (1+z_i u) \prod_{i=1}^l (1-t_i u), \\ \sum_{r=0}^{\infty} {}_k\vartheta_r^{(-l)}(x, z|t) \cdot u^r &= \prod_{i=1}^{\infty} \frac{1+x_i u}{1-x_i u} \prod_{i=1}^k (1+z_i u) \prod_{i=1}^l \frac{1}{1+t_i u}. \end{aligned}$$

For $r < 0$, we set ${}_k\vartheta_r^{(l)}(x, z|t) = 0$. We omit k when it is made clear by the context. Under π_n , the variables z_i correspond to the Chern roots of the tautological bundles and the theta polynomials ${}_k\vartheta_r^{(l)}(x, z|t)$ map to the Chern classes of certain virtual bundles (see Proposition 5.16). Although the above definition of the double theta polynomials appears slightly different from the one in Wilson's thesis [26] (also mentioned in [23]), one recovers Wilson's definition after applying appropriate changes of indices. See Remark 7.5 for details.

We show that $\mathcal{R}_\infty^{(k)}$ contains ${}_k\vartheta_r^{(l)}(x, z|t)$ (see §3.3) and can be described as follows (Corollary 6.10):

$$\mathcal{R}_\infty^{(k)} = \mathbb{Z}[t][\vartheta_1, \vartheta_2, \dots], \quad \vartheta_r := {}_k\vartheta_r^{(0)}. \quad (1.2)$$

Wilson proved the following fact.

Theorem 1.2 ([26]). *We have*

$$\mathfrak{C}_{w_r^{(k)}}(z, t; x) = {}_k\vartheta_r^{(r-k-1)}(x, z|t). \quad (1.3)$$

Let λ be a k -strict partition in $\mathcal{P}_n^{(k)}$. In the one-line notation of signed permutations (see §2), we can write the corresponding k -Grassmannian element $w_\lambda^{(k)}$ of $W_n^{(k)}$ as

$$\begin{aligned} w_\lambda^{(k)} &= v_1 v_2 \dots v_k | \overline{\zeta_1} \dots \overline{\zeta_s} u_1 \dots u_{n-k-s}, \\ 0 &< v_1 < \dots < v_k, \quad \overline{\zeta_1} < \dots < \overline{\zeta_s} < 0 < u_1 < \dots < u_{n-k-s}, \end{aligned} \quad (1.4)$$

where s is a non-negative integer. Let $\chi_\lambda = (\chi_1, \dots, \chi_{n-k})$ be the following sequence

$$\chi_\lambda := (\zeta_1 - 1, \dots, \zeta_s - 1, -u_1, \dots, -u_{n-k-s}) \in \mathbb{Z}^{n-k}. \quad (1.5)$$

We call χ_λ the *characteristic index* of λ . For a positive integer m , let $\Delta_m := \{(i, j) \mid 1 \leq i < j \leq m\}$. Define a subset $D(\lambda)$ of Δ_{n-k} by

$$D(\lambda) := \{(i, j) \in \Delta_{n-k} \mid \chi_i + \chi_j < 0\}. \quad (1.6)$$

We use the *multi Schur-Pfaffian* due to Kazarian [15], which is a natural variation of the Schur Pfaffian appearing in [22]. Let $(c^{(1)}, c^{(2)}, \dots, c^{(m)})$ be an m -tuple such that each $c^{(i)}$ is an infinite sequence of variables $c_r^{(i)}$ ($r \in \mathbb{Z}$). The multi Schur-Pfaffian $\text{Pf}[c_{r_1}^{(1)} c_{r_2}^{(2)} \dots c_{r_m}^{(m)}]$ is defined in §4. This is a finite \mathbb{Z} -linear combination of $c_{s_1}^{(1)} \dots c_{s_m}^{(m)}$, $(s_1, \dots, s_m) \in \mathbb{Z}^m$. For each $l = (l_1, \dots, l_m) \in \mathbb{Z}^m$, the substitution of $\vartheta_{s_i}^{(l_i)}$ to $c_{s_i}^{(i)}$ in this linear combination is denoted by

$$\text{Pf}[\vartheta_{r_1}^{(l_1)} \vartheta_{r_2}^{(l_2)} \dots \vartheta_{r_m}^{(l_m)}].$$

The main result of this paper is the following.

Theorem 1.3 (Pfaffian sum formula, Theorem 6.6 below). *Let λ be a k -strict partition in $\mathcal{P}_n^{(k)}$, and χ the corresponding characteristic index. We have*

$$\mathfrak{C}_{w_\lambda^{(k)}} = \sum_{I \subset D(\lambda)} \text{Pf} \left[\vartheta_{\lambda_1 + a_1^I}^{(\chi_1)} \cdots \vartheta_{\lambda_{n-k} + a_{n-k}^I}^{(\chi_{n-k})} \right], \quad (1.7)$$

where I runs over all subsets of $D(\lambda)$ and $a_s^I = \#\{j \mid (s, j) \in I\} - \#\{i \mid (i, s) \in I\}$.

Note that the right hand side does not depend on n , i.e., it depends only on $\lambda \in \mathcal{P}_\infty^{(k)}$. See Remark 6.4 for a more precise statement.

Example 1.4. Let $k = 1, n = 5$. Let $\lambda = (5, 3, 2, 1)$ be a k -strict partition. Then $w_\lambda^{(1)} = 5|4\bar{2}\bar{1}3$ and $D(\lambda) = \{(2, 4), (3, 4)\}$. We have

$$\mathfrak{C}_{w_{(5,3,2,1)}^{(1)}} = \text{Pf}[\vartheta_5^{(3)} \vartheta_3^{(1)} \vartheta_2^{(0)} \vartheta_1^{(-3)}] + \text{Pf}[\vartheta_5^{(3)} \vartheta_4^{(1)} \vartheta_2^{(0)} \vartheta_0^{(-3)}] + \text{Pf}[\vartheta_5^{(3)} \vartheta_3^{(1)} \vartheta_3^{(0)} \vartheta_0^{(-3)}].$$

Note that, even if $D(\lambda) \neq \emptyset$, it is possible that the double Schubert polynomial is a single Pfaffian in the formula. For example, we have $\mathfrak{C}_{13|\bar{5}\bar{4}2} = \text{Pf}[\vartheta_7^{(4)} \vartheta_5^{(2)} \vartheta_0^{(-4)}] + \text{Pf}[\vartheta_7^{(4)} \vartheta_4^{(2)} \vartheta_{-1}^{(-4)}]$ according to the formula, but the second term is zero (cf. Remark 5.8), and therefore $\mathfrak{C}_{13|\bar{5}\bar{4}2} = \text{Pf}[\vartheta_7^{(4)} \vartheta_5^{(2)} \vartheta_0^{(-4)}]$.

Once we read the formula in terms of raising operators, the following corollary is immediate.

Corollary 1.5. *If $D(\lambda) = \Delta_{n-k}$, in particular, if λ is contained in the $(n-k) \times k$ rectangle, then $\mathfrak{C}_{w_\lambda^{(k)}}$ is a single determinant*

$$\text{Det}[\vartheta_{\lambda_1}^{(\chi_1)} \cdots \vartheta_{\lambda_{n-k}}^{(\chi_{n-k})}] := \det(\vartheta_{\lambda_i + j - i}^{(\chi_i)})_{1 \leq i, j \leq n-k}. \quad (1.8)$$

If $D(\lambda) = \emptyset$, in particular, if λ is a strict partition containing the $(n-k) \times k$ rectangle, then $\mathfrak{C}_{w_\lambda^{(k)}}$ is a single Pfaffian

$$\text{Pf}[\vartheta_{\lambda_1}^{(\chi_1)} \cdots \vartheta_{\lambda_{n-k}}^{(\chi_{n-k})}].$$

The case when λ is contained in the $(n-k) \times k$ rectangle was considered by Wilson. She calls such partition a “small” partition and proved in [26, Section 5.8] that the corresponding double theta polynomial, written as the determinant (1.8), satisfies an appropriate vanishing property.

It is straightforward to apply our results to the problem of degeneracy loci formulas of vector bundles (cf. [1], [2], [23]). As the simplest example, we provide a Chern class interpretation of ${}_k\vartheta_r^{(l)}(z, x|t)$ in §5.3. It is worth mentioning that the special cases appearing in Corollary 1.5 look precisely the same as the classical Kempf-Laksov determinantal formula for type A degeneracy loci [14] and the Pfaffian formula for Lagrangian degeneracy loci [10], [15] (see also Remark 4.2), although the functions are associated to the isotropic Grassmannians.

In the proof of Theorem 1.3, the left divided difference operators δ_i play an essential role. By a direct calculation, we show that the right hand side of (1.7) satisfies the part of the defining properties of the double Schubert polynomials corresponding to the left divided difference operators. To finish the proof, we then make use of a uniqueness lemma (Lemma 3.11) which is available for the parabolic case.

1.5. Beyond Grassmannians. The method described above to derive the Pfaffian sum formula for double Schubert polynomials works beyond the k -Grassmannian elements. Namely, we first derive a single Pfaffian formula for the top class for symplectic partial flag variety (Theorem 8.10). Then we introduce a certain family of signed permutations (*pseudo k -Grassmannian elements*, see Definition 8.12) and derive the Pfaffian sum formula for them (Theorem 8.13). Those signed permutations form a subset of the minimum-length coset representatives for non-maximal parabolic cases, *i.e.* they correspond to certain Schubert varieties of the isotropic partial flag varieties. In this case, one can regard the entries of the Pfaffians as special classes arising from various isotropic Grassmannians.

Furthermore, we can go beyond the pseudo k -Grassmannian elements. For example, among all 48 signed permutations in W_3 , 16 of them are not pseudo k -Grassmannian elements. It turns out that those 16 are also written as (sums of) Pfaffians, *except* $\bar{3}2\bar{1}$, $\bar{3}21$, $\bar{2}31$ and $\bar{1}32$. It would be interesting to study in general to what extent the double Schubert polynomials are written as sums of Pfaffians, and also the geometric or combinatorial conditions for the polynomials to be Pfaffian sums.

1.6. Related results. Anderson and Fulton [2] defined a notion of vexillary signed permutation in type B, C, and D. They showed that the double Schubert polynomials associated to vexillary signed permutations are given by explicit Pfaffian formulas. Naruse [19] also independently proved a formula that expresses the corresponding double Schubert polynomials as a specialization of the factorial Q - and P -functions. Since our formula also express some Schubert classes as single Pfaffians, there is an overlap between our results and the results of [2], [19]. However, not all (pseudo) k -Grassmannian permutations are vexillary and there are non-vexillary k -Grassmannian permutations whose corresponding classes are written as single Pfaffians, *e.g.* $13|\bar{5}42$ is not vexillary but $\mathfrak{C}_{13|\bar{5}42}$ is a single Pfaffian as above.

Tamvakis [24] proved a combinatorial formula which expresses an arbitrary (equivariant) Schubert class of any classical G/P space as a polynomial in the special Schubert classes (see also [23]). The formula involves some combinatorial data related to the reduced decomposition of Weyl group elements as well as theta polynomials and Schur S -functions.

Beside the possibility of extending our methods and results to type D, it is also natural to ask if our formula can be derived by using Kazarian's pushforward formula. If it is possible, there will arise a new perspective hopefully applicable to K-theory case. We hope to address these problems elsewhere.

Note added. – After the submission of this manuscript, we have received from Harry Tamvakis a preprint [25] written by him and Wilson, where they provided a presentation of the equivariant cohomology ring of the symplectic Grassmannian. Some of their arguments use our main result in an essential way.

1.7. Organization. This paper is organized as follows. In Section 2, we review the double Schubert polynomials (DSP) following [11]. In Section 3, we give some preliminary discussions on the symplectic Grassmannian. In Section 4, we introduce the multi Schur-Pfaffian used by Kazarian in a slightly generalized form. In Section 5, we introduce the double theta polynomials and establish some of their basic properties. Section 6 is devoted to the proof of Theorem 1.3. In Section 7, we introduce the raising operators and their action on formal power series to prove the equivalence of our main theorem and the conjecture appearing in Wilson's thesis [26]. In Section

8, we prove the Pfaffian sum formula for pseudo k -Grassmannian permutations. In Section 9 for the cases $(n, k) = (5, 2), (5, 3)$, we provide the expressions for the k -Grassmannian elements.

Acknowledgements: We are especially grateful to Hiroshi Naruse for explaining his results, and also to Harry Tamvakis for valuable comments to an earlier version of this manuscript. We thank Dave Anderson, Anders Skovsted Buch, Andrew Kresch, Changzheng Li, Leonardo Mihalcea, Masaki Nakagawa for the helpful conversations and their comments. We thank the anonymous referee and Harry Tamvakis for independently pointing out an error of an argument in proving Theorem 4 in a previous version. We also thank Thomas Hudson for carefully reading the manuscript. This paper was written for the most part during the first named author's stay at KAIST in 2013. The hospitality and perfect working conditions there are gratefully acknowledged.

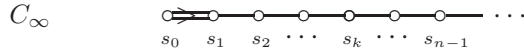
2. DOUBLE SCHUBERT POLYNOMIALS OF TYPE C

In this section, we review the construction of the double Schubert polynomials, following [11]. The expository article [23] by Tamvakis will be also helpful to grasp more geometric backgrounds of this construction.

Let W_∞ be the group defined by the generators $\{s_i \mid i = 0, 1, \dots\}$ and the relations

$$\begin{aligned} s_i^2 &= e \quad (i \geq 0), & s_0 s_1 s_0 s_1 &= s_1 s_0 s_1 s_0, & s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} \quad (i \geq 1), \\ s_i s_j &= s_j s_i \quad (|i - j| \geq 2). \end{aligned}$$

The corresponding Dynkin diagram is the following.



The group W_∞ is identified with the set of all permutations w of the set $\{1, 2, \dots\} \cup \{\bar{1}, \bar{2}, \dots\}$ such that $w(i) \neq i$ for only finitely many i , and $\overline{w(i)} = w(\bar{i})$ for all i . The simple reflections are identified with the transpositions $s_0 = (1, \bar{1})$ and $s_i = (i+1, i)(\bar{i}, \bar{i}+1)$ for $i \geq 1$. The Weyl group W_n is identified with the subgroup of W_∞ consisting of w such that $w(i) = i$ for all $i > n$. In one-line notation, we often denote an element $w \in W_n$ by the finite sequence $(w(1), \dots, w(n))$.

The function $Q_r(x)$ is defined by the generating function

$$\sum_{r=0}^{\infty} Q_r(x) u^r = \prod_{i=1}^{\infty} \frac{1 + x_i u}{1 - x_i u} \quad (2.1)$$

i.e., $Q_r(x) = {}_0\vartheta_r^{(0)}$. Let Γ be the ring ¹ generated by $Q_r(x)$ ($r \geq 1$). Let \mathcal{R}_∞ be the polynomial ring $\Gamma[t, z]$ in the variables $t = (t_1, t_2, \dots)$, and $z = (z_1, z_2, \dots)$ with coefficients in Γ . There are two actions of W_∞ on the ring \mathcal{R}_∞ defined below. We denote the corresponding operators on \mathcal{R}_∞ by s_i^z (*right action*) and s_i^t (*left action*).

¹ It is well-known that Γ can be defined as the quotient of the polynomial ring $\mathbb{Z}[Q_1, Q_2, Q_3, \dots]$ of the variables Q_1, Q_2, \dots by the ideal generated by the following elements

$$Q_r^2 + 2 \sum_{i=1}^r (-1)^i Q_{r+i} Q_{r-i} \quad (r \geq 1), \quad (2.2)$$

with $Q_0 = 1$. This fact follows from [17, (8.6) (ii), (8.2')] and Proof of (8.4) in Chap. III].

For $i \geq 1$, let $s_i^z(z_i) = z_{i+1}$, $s_i^z(z_{i+1}) = z_i$, $s_i^z(z_j) = z_j$ ($j \neq i, i+1$), and $s_i^z(Q_r(x)) = Q_r(x)$. There is an automorphism s_0^z of $\mathbb{Z}[t]$ -algebra on \mathcal{R}_∞ characterized by the following:

$$s_0^z(z_1) = -z_1, \quad s_0^z(z_i) = z_i \ (i \geq 1), \quad \sum_{r=0}^{\infty} s_0^z(Q_r(x))u^r = \frac{1+z_1u}{1-z_1u} \prod_{i=1}^{\infty} \frac{1+x_iu}{1-x_iu}.$$

The last equation is equivalently written as

$$s_0^z Q_r(x_1, x_2, \dots) = Q_r(z_1, x_1, x_2, \dots).$$

Clearly we can extend s_i^z to \mathcal{R}_∞ as an automorphism of $\mathbb{Z}[t]$ -algebra and show that $s_i \mapsto s_i^z$ ($i \geq 0$) gives a right action of W_∞ on \mathcal{R}_∞ . Similarly, there are operators s_i^t ($i \geq 0$) on \mathcal{R}_∞ such that $s_i \mapsto s_i^t$ ($i \geq 0$) gives a left action of W_∞ on \mathcal{R}_∞ as $\mathbb{Z}[z]$ -algebra automorphisms. In order to define this action, we can use the following ring automorphism ω :

$$\omega(t_i) = -z_i, \quad \omega(z_i) = -t_i, \quad \omega(Q_r(x)) = Q_r(x).$$

We define $s_i^t = \omega s_i^z \omega$ ($i \geq 0$). In particular, we have

$$s_0^t Q_r(x_1, x_2, \dots) = Q_r(-t_1, x_1, x_2, \dots).$$

Define the *simple roots* by

$$\alpha_0 = 2t_1, \quad \alpha_i = t_{i+1} - t_i \ (i \geq 1).$$

The right and left divided difference operators are defined by

$$\partial_i f = \frac{f - s_i^z f}{\omega(\alpha_i)}, \quad \delta_i f = \frac{f - s_i^t f}{\alpha_i} \quad (i \geq 0, f \in \mathcal{R}_\infty).$$

Theorem 2.1 ([11]). *There exists a unique $\mathbb{Z}[t]$ -free basis $\{\mathfrak{C}_w(z, t; x) \mid w \in W_\infty\}$ of \mathcal{R}_∞ satisfying the equations*

$$\partial_i \mathfrak{C}_w = \begin{cases} \mathfrak{C}_{ws_i} & \text{if } \ell(ws_i) < \ell(w), \\ 0 & \text{otherwise,} \end{cases} \quad \delta_i \mathfrak{C}_w = \begin{cases} \mathfrak{C}_{s_i w} & \text{if } \ell(s_i w) < \ell(w), \\ 0 & \text{otherwise,} \end{cases} \quad (2.3)$$

for all $i \geq 0$, and such that \mathfrak{C}_w has no constant term except for $\mathfrak{C}_e = 1$.

3. PRELIMINARIES ON THE SYMPLECTIC GRASSMANNIAN

3.1. k -strict partitions. We develop some combinatorics related to the Schubert classes of SG_n^k . The set of the minimum-length coset representatives for $W_\infty/W_{(k)}$ is given by

$$W_\infty^{(k)} = \{w \in W_\infty \mid \ell(w) > \ell(ws_i) \ (\forall i \geq 0, i \neq k)\}.$$

We will review the bijection $\mathcal{P}_\infty^{(k)} \rightarrow W_\infty^{(k)}$ ($\lambda \mapsto w_\lambda^{(k)}$) such that $|\lambda| = \ell(w_\lambda^{(k)})$, which is due to [4] (cf. [23, Section 4.2]). Each $w \in W_\infty^{(k)}$ is called a k -Grassmannian permutation, and if $w \in W_n$, we can write w in the one-line notation

$$w = v_1 \cdots v_k | \overline{\zeta_1} \cdots \overline{\zeta_s} u_1 \cdots u_{n-k-s},$$

as in (1.4). For each i with $1 \leq i \leq k$, let μ_i be the number of the elements of $\{u_1, \dots, u_{n-k-s}\}$ less than v_{k+1-i} . Then $\mu = (\mu_1, \dots, \mu_k)$ is a partition whose Young diagram fits inside the $k \times (n-s-k)$ rectangle. Let ν be the conjugate of μ (the transpose of the Young diagram). It is worth noting that

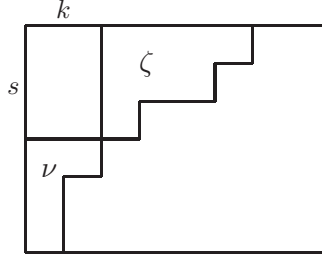
$$\nu_i = k - \#\{p \mid v_p < u_i\} = \#\{p \mid v_p > u_i\} \quad (i = 1, \dots, n-k-s). \quad (3.1)$$

Consider the strict partition $\zeta := (\zeta_1, \dots, \zeta_s)$ defined by the entries with bars in the one-line notation. The k -strict partition λ corresponding to the k -Grassmannian permutation w is given by

$$\lambda_i = \begin{cases} \zeta_i + k & \text{if } 1 \leq i \leq s, \\ \nu_{i-s} & \text{if } s+1 \leq i \leq n-k. \end{cases} \quad (3.2)$$

Note that the k -strict partition $\lambda \in \mathcal{P}_\infty^{(k)}$ defined from $w \in W_\infty^{(k)}$ as above is independent of the choice of n such that $w \in W_n$.

Example 3.1. The 2-Grassmannian permutation $w = 58|4\bar{3}\bar{1}267$ corresponds to the 2-strict partition $\lambda = (6, 5, 3, 2, 1, 1)$.



On the other hand, we can reconstruct $w_\lambda^{(k)}$ from a k -strict partition $\lambda = (\lambda_1, \dots, \lambda_r > 0)$ in $\mathcal{P}_\infty^{(k)}$. Let $\lambda_1, \dots, \lambda_s > k$ and $\lambda_{s+1} \leq k$. Let $\mu = (\mu_1, \dots, \mu_k)$ be the conjugate of the partition $(\lambda_{s+1}, \dots, \lambda_r)$. Define $\zeta_i := \lambda_i - k$ for $i = 1, \dots, s$. Define (v_1, \dots, v_k) by

$$v_{k+1-i} = \mu_i + s + k + 1 - i - \#\{\zeta_j \mid \zeta_j \geq \mu_i + s + k + 1 - i\}.$$

The signed permutation $w_\lambda^{(k)}$ is given by

$$w_\lambda^{(k)} = (v_1 \dots v_k | \bar{\zeta}_1, \dots, \bar{\zeta}_s u_1 u_2, \dots) \in W_\infty^{(k)}$$

where u_1, u_2, \dots form an increasing sequence of positive integers, which is determined uniquely by the integers v_i and ζ_i . Note that

$$u_j = j + \#\{p \mid v_p < u_j\} + \#\{\zeta_p \mid \zeta_p < u_j\}. \quad (3.3)$$

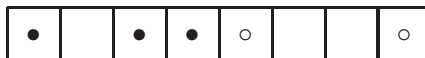
Since $\zeta_i > u_j$ if and only if $\#\{\zeta_p \mid \zeta_p > u_j\} > i$, (3.3) implies that

$$\zeta_i > u_j \text{ if and only if } \zeta_i > j + \#\{p \mid v_p < u_j\} + s - i. \quad (3.4)$$

Remark 3.2. If $k > 0$, the partial order on $\mathcal{P}_\infty^{(k)}$ given by the inclusion of the Young diagrams is not compatible with the one on $W_\infty^{(k)}$ induced from the Bruhat order on W_∞ . For example, if we let $k = 2$, $\lambda = (3, 2)$, $\mu = (5, 1)$, then we have $w_\lambda^{(2)} = 34\bar{1}2 \dots = s_2 s_0 s_1 s_3 s_2$ and $w_\mu^{(2)} = 14\bar{3}2 \dots = s_2 s_1 s_0 s_1 s_3 s_2$. So we have $w_\lambda^{(2)} \leq w_\mu^{(2)}$ in the Bruhat order, but $\lambda \not\leq \mu$.

It would be an interesting problem to give a good combinatorial model for $W_\infty^{(k)}$ which enable us to see the Bruhat order manifestly. One candidate is the Maya diagram introduced below.

Remark 3.3. We can depict the permutation w as the following “Maya diagram”.



The integers v_1, \dots, v_k are the positions of the boxes with \circ , while ζ_1, \dots, ζ_s are the positions of the boxes with \bullet . Then μ_i is the number of the vacant boxes to the left of the i th box with \circ . In the above diagram, we have $\zeta = (4, 3, 1)$ and $\mu = (3, 1)$, so $\nu = (2, 1, 1)$.

We record the following lemmas without proofs and will use them later in the proof of the main theorem. (cf. [3, Proposition 8.1.1]).

Lemma 3.4. *Let $w = v_1 \cdots v_k | \overline{\zeta_1} \cdots \overline{\zeta_s} u_1 \cdots u_{n-k-s} \in W_n^{(k)}$. Suppose $i \geq 1$. $\ell(s_i w) < \ell(w)$ if and only if one of the following holds:*

- (L1) $w = (\cdots | \cdots \overline{i+1} \cdots i \cdots)$, i.e., $\zeta_p = i+1$ and $u_q = i$ for some p and q ;
- (L2) $w = (\cdots i \cdots | \cdots \overline{i+1} \cdots)$, i.e., $\zeta_p = i+1$ and $v_q = i$ for some p and q ;
- (L3) $w = (\cdots i+1 \cdots | \cdots i \cdots)$, i.e., $u_p = i$ and $v_q = i+1$ for some p and q ;

Note that, in this case, $s_i w \in W_\infty^{(k)}$.

Lemma 3.5. *Let $w = v_1 \cdots v_k | \overline{\zeta_1} \cdots \overline{\zeta_s} u_1 \cdots u_{n-k-s} \in W_n^{(k)}$.*

- (L0) $\ell(s_0 w) < \ell(w)$ if and only if $w = (\cdots | \cdots \bar{1} \cdots)$, i.e., $\zeta_s = 1$.

3.2. Remarks on the Schubert conditions. In this section, we review the definition of Schubert classes of SG_n^k for the sake of the precise comparison of our conventions and those in [11]. See also [7, Section 6] and [4, Section 0]. It is worth noting that the characteristic index χ appears in the Schubert conditions in an apparent manner.

Let $\mathbf{e}_1, \dots, \mathbf{e}_n, \mathbf{e}_1^*, \dots, \mathbf{e}_n^*$ be a standard symplectic basis of \mathbb{C}^{2n} . Define a symplectic form by

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \langle \mathbf{e}_i^*, \mathbf{e}_j^* \rangle = 0, \quad \langle \mathbf{e}_i, \mathbf{e}_j^* \rangle = \delta_{ij}.$$

For $1 \leq i \leq n$, define a complete flag $F^\bullet : F^n \subset \cdots \subset F^1 \subset F^{\bar{1}} \subset \cdots \subset F^{\bar{n}}$ by

$$F^i = \langle \mathbf{e}_i, \dots, \mathbf{e}_n \rangle, \quad F^{\bar{i}} = \langle \mathbf{e}_i^*, \dots, \mathbf{e}_1^* \rangle + F^1.$$

Let $\lambda \in \mathcal{P}_n^{(k)}$. Then the Schubert variety Ω_λ with respect to F^\bullet can be also defined as

$$\Omega_\lambda := \{V \in SG_n^k \mid \dim(V \cap \overline{F^{w_\lambda^{(k)}(k+j)}}) \geq j \quad (1 \leq j \leq n-k)\}. \quad (3.5)$$

Indeed, if we relabel the above flag by $F_1 \subset \cdots \subset F_{2n}$, i.e., $F^i = F_{n+1-i}$ and $F^{\bar{i}} = F_{n+i}$ for $1 \leq j \leq n$, then

$$\overline{F^{w_\lambda^{(k)}(k+j)}} = \begin{cases} F^{\zeta_j} = F_{n+1-\zeta_j} = F_{n-\chi_j} & \text{if } 1 \leq j \leq s, \\ \overline{F^{u_{j-s}}} = F_{n+u_{j-s}} = F_{n-\chi_j} & \text{if } s+1 \leq j \leq n-k, \end{cases}$$

where $v_1 \cdots v_k | \overline{\zeta_1} \cdots \overline{\zeta_s} u_1 \cdots u_{n-k-s}$ is the one-line notation of $w_\lambda^{(k)}$. Therefore the equivalence of the definitions of Ω_λ at (1.1) and here follows from

$$p_j(\lambda) = n - \chi_j \quad (1 \leq j \leq \ell(\lambda)) \quad (3.6)$$

and the fact that the condition is redundant for $j > \ell(\lambda)$. We can prove the equation (3.6) as follows. If $1 \leq j \leq s$, then the RHS is $n - p_j(\lambda) = \zeta_j - 1 = \chi_j$. Suppose that $s+1 \leq j \leq \ell(\lambda)$.

Then from the correspondence in §3.1, it is clear that

$$\begin{aligned}
-\chi_j = u_{j-s} &= j - s + \#\{p \mid v_p < u_{j-s}\} + \#\{\zeta_i \mid \zeta_i < u_{j-s}\} \\
&= k - \lambda_j + 1 + j - s - 1 + \#\{\zeta_i \mid \zeta_i < u_{j-s}\} \\
&= k - \lambda_j + 1 + \#\{i \mid i < j, \chi_i + \chi_j \leq -1\} \\
&= k - \lambda_j + 1 + j - 1 - \#\{i \mid i < j, \chi_i + \chi_j \geq 0\} \\
&= k - \lambda_j + 1 + j - 1 - \#\{i \mid i < j, \lambda_i + \lambda_j > 2k + j - i\} \\
&= p_j(\lambda) - n.
\end{aligned}$$

The first equality follows from (3.3), the second equality follows from (3.1) and (3.2), and the second last equality follows from the following lemma.

Lemma 3.6. *Let χ be the characteristic index of λ . Suppose $1 \leq i < j \leq n - k$. Then $\chi_i + \chi_j \geq 0$ if and only if $\lambda_i + \lambda_j > 2k + j - i$.*

Proof. Let $v_1 \dots v_k | \overline{\zeta_1} \dots \overline{\zeta_s} u_1 \dots u_{n-k-s}$ be the one-line notation for $w_\lambda^{(k)}$. The only non-trivial case is when $i \leq s$ and $j \geq s + 1$. The equivalence (3.4) implies that $\chi_i + \chi_j = \zeta_i - 1 - u_{j-s} \geq 0$ if and only if

$$\lambda_i + \lambda_j = 2k + \zeta_i - \#\{p \mid v_p < u_{j-s}\} > 2k + j - i$$

where the first equality follows from (3.1) and (3.2). \square

The T -fixed point of SG_n^k corresponding to λ is $\langle \mathbf{e}_{w_\lambda^{(k)}(k+1)}^*, \dots, \mathbf{e}_{w_\lambda^{(k)}(n)}^* \rangle$, which is the image of $e_w \in \mathcal{F}l_n$ under the projection pr_k onto SG_n^k .

3.3. Invariant subring $\mathcal{R}_\infty^{(k)}$. Let $\mathcal{R}_\infty^{(k)}$ be the subring of elements in \mathcal{R}_∞ which are fixed by the right action of $W_{(k)}$:

$$\mathcal{R}_\infty^{(k)} := \{f \in \mathcal{R}_\infty \mid s_i^z(f) = f \quad (\forall i \neq k)\}.$$

Since the right action of W_∞ is $\mathbb{Z}[t]$ -linear, $\mathcal{R}_\infty^{(k)}$ is a $\mathbb{Z}[t]$ -subalgebra of \mathcal{R}_∞ .

Proposition 3.7. *We have*

$$\mathcal{R}_\infty^{(k)} = \bigoplus_{w \in W_\infty^{(k)}} \mathbb{Z}[t] \mathfrak{C}_w.$$

Proof. In order to prove the inclusion “ \supset ”, it is enough to show $\mathfrak{C}_w \in \mathcal{R}_\infty^{(k)}$ for all $w \in W_\infty^{(k)}$. Let $w \in W_\infty^{(k)}$. Then for any $j \neq k$, we have $\ell(ws_j) = \ell(w) + 1$, and hence $\partial_j \mathfrak{C}_w = 0$. This is equivalent to $s_j^z \mathfrak{C}_w = \mathfrak{C}_w$ for $j \neq k$. Thus we have $\mathfrak{C}_w \in \mathcal{R}_\infty^{(k)}$. To prove the reverse inclusion “ \subset ”, we write an arbitrary element f of $\mathcal{R}_\infty^{(k)}$ as $f = \sum_{w \in W_\infty} c_w \mathfrak{C}_w$ ($c_w \in \mathbb{Z}[t]$). If $v \notin W_\infty^{(k)}$, then there is i such that $i \neq k$ and $\ell(vs_i) < \ell(v)$. We have $0 = \partial_i f = \sum_{v \in W_\infty, \ell(vs_i) = \ell(v)-1} c_v \mathfrak{C}_{vs_i}$. It follows that $c_v = 0$. \square

Definition 3.8. For each $\mu \in \mathcal{P}_\infty^{(k)}$, let $w_\mu^{(k)} = v_1 v_2 \dots v_k | \overline{\zeta_1} \overline{\zeta_2} \dots \overline{\zeta_s} u_1 u_2 \dots \in W_\infty^{(k)}$ be the corresponding signed permutation. Define

$$\Phi : \mathcal{R}_\infty^{(k)} \rightarrow \text{Map}(\mathcal{P}_\infty^{(k)}, \mathbb{Z}[t]); \quad f \mapsto (\mu \mapsto f|_\mu),$$

where $f|_\mu$ is defined by the substitution

$$\begin{aligned}
(z_1, \dots, z_k) &\mapsto (t_{v_1}, \dots, t_{v_k}); \\
(x_1, x_2, \dots) &\mapsto (t_{\zeta_1}, \dots, t_{\zeta_s}, 0, 0, \dots).
\end{aligned}$$

Remark 3.9. Note that this is the restriction of the universal localization map defined in [11, Section 6.1]. In particular, we have the following vanishing property:

$$\mathfrak{C}_{w_\lambda^{(k)}}|_\mu = 0 \quad \text{unless } w_\lambda^{(k)} \leq w_\mu^{(k)}. \quad (3.7)$$

Here \leq is the Bruhat-Chevalley order.

Lemma 3.10. The homomorphism $\Phi : \mathcal{R}_\infty^{(k)} \rightarrow \text{Map}(\mathcal{P}_\infty^{(k)}, \mathbb{Z}[t])$ is injective.

Proof. The proof is identical to the one for [11, Lemma 6.5]. \square

Lemma 3.11. If a family $F_w, w \in W_\infty^{(k)}$ of elements of $\mathcal{R}_\infty^{(k)}$ satisfies the following conditions

$$\delta_i F_w = \begin{cases} F_{s_i w} & \text{if } \ell(s_i w) < \ell(w), \\ 0 & \text{if } \ell(s_i w) > \ell(w), \end{cases} \quad (3.8)$$

$$F_w|_\emptyset = \delta_{w,e}, \quad (3.9)$$

then $F_w = \mathfrak{C}_w$ for all $w \in W_\infty^{(k)}$.

Proof. First note that the family $\mathfrak{C}_w, w \in W_\infty^{(k)}$ satisfies those relations: (3.8) by Theorem 2.1, and (3.9) by the vanishing property (3.7) and the fact that $\mathfrak{C}_e = 1$ (Theorem 2). By Lemma 3.10, it suffices to show that, for each $w \in W_\infty^{(k)}$, the localization $(F_w - \mathfrak{C}_w)|_\mu$ is zero for all $\mu \in \mathcal{P}_\infty^{(k)}$. We use the induction on $|\mu|$. By (3.9), we have $(F_w - \mathfrak{C}_w)|_\emptyset = 0$ for every $w \in W_\infty^{(k)}$. Now assume that $\mu \neq \emptyset$ and that for each $w \in W_\infty^{(k)}$, the localization $(F_w - \mathfrak{C}_w)|_{\mu'}$ is zero for all $\mu' \in \mathcal{P}_\infty^{(k)}$ such that $|\mu'| < |\mu|$. Since $\mu \neq \emptyset$, there is $i \geq 0$ such that $s_i w_\mu^{(k)} < w_\mu^{(k)}$. This implies that $s_i w_\mu^{(k)}$ is a minimum-length coset representative, i.e. an element of $W_\infty^{(k)}$. Therefore $s_i w_\mu^{(k)} = w_\nu^{(k)}$ for some $\nu \in \mathcal{P}_\infty^{(k)}$ with $|\nu| = |\mu| - 1$. By the definition of δ_i and the localization at μ , the equations (3.8) implies the following recurrence relation

$$F_w|_\mu = \begin{cases} s_i(F_w|_\nu) + \alpha_i \cdot s_i(F_{s_i w}|_\nu) & \text{if } \ell(s_i w) < \ell(w), \\ s_i(F_w|_\nu) & \text{if } \ell(s_i w) > \ell(w). \end{cases}$$

Since \mathfrak{C}_w also satisfies the same recurrence relation, the difference $(F_w - \mathfrak{C}_w)|_\mu$ vanishes by the induction hypothesis. Thus $F_w = \mathfrak{C}_w$ for every $w \in W_\infty^{(k)}$. \square

3.4. Duality of SG_n^k . There is a unique longest element in $W_n^{(k)}$, which we denote by w_{\max} . In the one-line notation, it is given by $12 \cdots k | \overline{n} \cdots \overline{n-k}$. For $w \in W_n^{(k)}$, define $w^\vee = w w_{\max}$. We have $w^\vee \in W_n^{(k)}$ and $\ell(w^\vee) = \ell(w_{\max}) - \ell(w)$. Moreover, we have $w_{\max}^2 = e$, and so the operation $w \mapsto w^\vee$ is an involution on the set $W_n^{(k)}$. Note that this involution *does* depend on n .

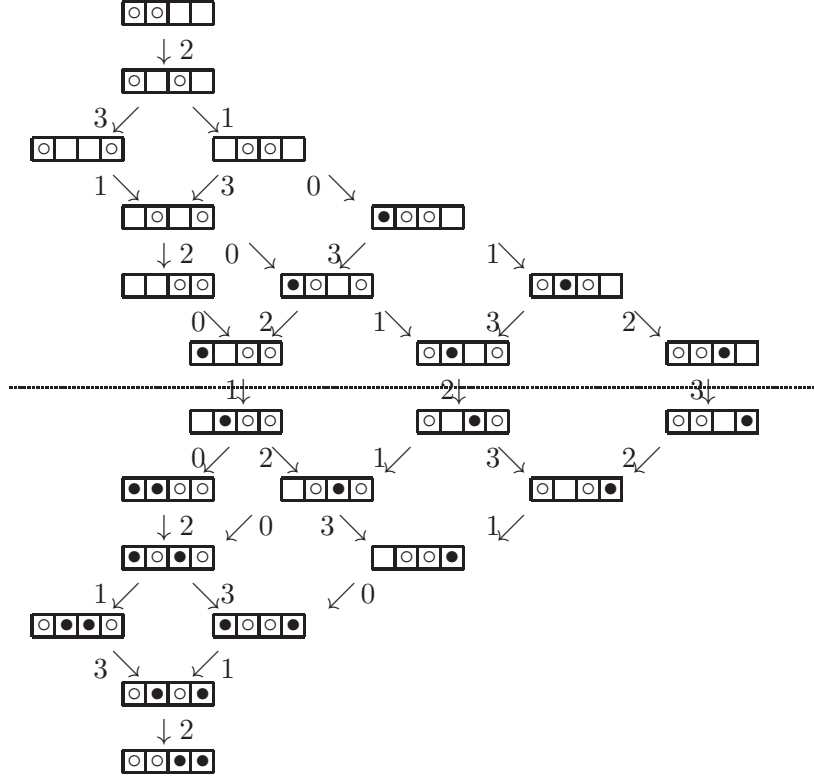
Remark 3.12. If $w = v_1 \cdots v_k | \overline{\zeta_1} \cdots \overline{\zeta_s} u_1 \cdots u_{n-k-s}$, then

$$w^\vee = v_1 \cdots v_k | \overline{u_{n-k-s}} \cdots \overline{u_1} \zeta_s \cdots \zeta_1.$$

In other words, the involution in terms of Maya diagram is given by exchanging the vacant boxes and the boxes occupied by “•”.

Let $w, v \in W_n^{(k)}$ and $i \in \mathcal{I} := \{0, 1, \dots, n-1\}$. We write $w \xrightarrow{i} v$ if $s_i w = v$ and $\ell(v) = \ell(w) + 1$. The relation is called the covering relation for the weak left Bruhat order ([3]). The *weak Bruhat graph* is the graph such that the set of vertices as $W_n^{(k)}$ and the (oriented) arrows are the covering relation for the weak left Bruhat order.

Example 3.13. Let $n = 4$, $k = 2$. We can draw the weak Bruhat graph as follows. The involution is given by reflection with respect to the dashed horizontal line.



Let $w \in W_n^{(k)}$. Define the following sets:

$$\begin{aligned} \mathcal{I}_-(w) &:= \{i \in \mathcal{I} \mid \ell(s_i w) = \ell(w) - 1\}, \\ \mathcal{I}_+(w) &:= \{i \in \mathcal{I} \mid \ell(s_i w) = \ell(w) + 1 \text{ and } s_i w \in W_n^{(k)}\}, \\ \mathcal{I}_0(w) &:= \{i \in \mathcal{I} \mid \ell(s_i w) = \ell(w) + 1 \text{ and } s_i w \notin W_n^{(k)}\}. \end{aligned} \quad (3.10)$$

Note that if $i \in \mathcal{I}_-(w)$ then $s_i w \in W_n^{(k)}$.

Example 3.14. Let $n = 4, k = 2$. If $w = 23\bar{4}1$. Then $\mathcal{I}_-(w) = \{1, 3\}$, $\mathcal{I}_+(w) = \{0\}$, and $\mathcal{I}_0(w) = \{2\}$.

Lemma 3.15. Let $w \in W_n^{(k)}$. Then the following hold.

- (1) $\mathcal{I}_-(w) = \mathcal{I}_+(w^\vee)$, $\mathcal{I}_+(w) = \mathcal{I}_-(w^\vee)$,
- (2) $\mathcal{I}_0(w) = \mathcal{I}_0(w^\vee)$.

Proof. (1) We will show $\mathcal{I}_-(w) \subset \mathcal{I}_+(w^\vee)$. Let $i \in \mathcal{I}_-(w)$. Then $s_i w \in W_n^{(k)}$ as noted above. Since $(s_i w)^\vee = s_i w w_{\max} = s_i w^\vee \in W_n^{(k)}$, we have

$$\ell(s_i w^\vee) = \ell((s_i w)^\vee) = \ell(w_{\max}) - \ell(s_i w) = \ell(w_{\max}) - (\ell(w) - 1) = \ell(w^\vee) + 1.$$

Thus $i \in \mathcal{I}_+(w^\vee)$. The proof of the opposite inclusion is similar. The second statement follows from the fact $(w^\vee)^\vee = w$.

- (2) $\mathcal{I}_0(w)$ is the complement of $\mathcal{I}_-(w) \cup \mathcal{I}_+(w)$ in \mathcal{I} . Hence the result follows from (1). \square

Lemma 3.16. Let $w \in W_n^{(k)}$. If $i \in \mathcal{I}_0(w)$, then $s_i w = w s_j$ for some $j (\neq k)$.

Proof. Because $s_i w \notin W_n^{(k)}$, there exists $j (\neq k)$ such that $\ell(s_i w s_j) = \ell(s_i w) - 1$. This means that $s_i w$ has a reduced expression of a form $s_{i_1} \cdots s_{i_l} s_j$ with $l = \ell(w)$ (cf. [3, Cor. 1.4.6.]) Since $\ell(s_i w) = \ell(w) + 1$, w is obtained from the reduced expression of $s_i w$ by deleting one *unique* simple reflection (“exchange condition” cf. [9, p.117]). Now the right end s_j is the unique one to be deleted since, otherwise, it contradicts with the assumption $w \in W_n^{(k)}$. Thus $w = s_{i_1} \cdots s_{i_l}$, and the lemma follows. \square

3.5. Lemma on left divided difference operators. For $w \in W_\infty$, we choose a reduced expression $s_{i_1} \cdots s_{i_l}$ for w . Then $\delta_w := \delta_{i_1} \cdots \delta_{i_l}$ does not depend on the reduced expression. The following fact is well-known (see for example [18, §2]).

Lemma 3.17. *Let $u, v \in W$. Then*

$$\delta_u \delta_v = \begin{cases} \delta_{uv} & \text{if } \ell(u) + \ell(v) = \ell(uv), \\ 0 & \text{if } \ell(u) + \ell(v) > \ell(uv). \end{cases}$$

The following proposition will be used in the proof of the main theorem.

Proposition 3.18. *Let $w \in W_n^{(k)}$. We have the following.*

- (1) *If $i \in \mathcal{I}_-(w)$ then $\delta_i \delta_{w^\vee} = \delta_{(s_i w)^\vee}$.*
- (2) *If $i \in \mathcal{I}_+(w)$ then $\delta_i \delta_{w^\vee} = 0$.*
- (3) *If $i \in \mathcal{I}_0(w)$ then there exists $j \neq k$ such that $\delta_i \delta_{w^\vee} = \delta_{w^\vee} \delta_j$.*

Proof. (1) If $i \in \mathcal{I}_-(w)$, then from Lemma 3.15, we have $i \in \mathcal{I}_+(w^\vee)$, i.e., $s_i w^\vee \in W_n^{(k)}$ and $\ell(s_i w^\vee) = \ell(w^\vee) + 1$. Recall that $s_i w^\vee = (s_i w)^\vee$. Then the result follows from Lemma 3.17.

(2) If $i \in \mathcal{I}_+(w)$, then from Lemma 3.15, we have $i \in \mathcal{I}_-(w^\vee)$. This means that $\ell(s_i w^\vee) = \ell(w^\vee) - 1$. Hence $\delta_{s_i w^\vee} = 0$ by Lemma 3.17.

(3) If $i \in \mathcal{I}_0(w)$, then from Lemma 3.15, we have $i \in \mathcal{I}_0(w^\vee)$. Then from Lemma 3.16 there exists some $j \neq k$ such that $s_i w^\vee = w^\vee s_j$, where the products in both hand sides are length-additive. Then the result follows from Lemma 3.17. \square

4. MULTI SCHUR-PFAFFIAN

In this section, we recall the multi Schur-Pfaffian due to Kazarian, but in a slightly more general form.

Let $(c^{(1)}, c^{(2)}, \dots, c^{(m)})$ be an m -tuple such that each $c^{(i)}$ is an infinite sequence of variables $c_r^{(i)}$ ($r \in \mathbb{Z}$). For any $(r_1, \dots, r_m) \in \mathbb{Z}^m$, the multi Schur-Pfaffian

$$\text{Pf}[c_{r_1}^{(1)} \cdots c_{r_m}^{(m)}] \in \mathbb{Z}[c_r^{(1)}, c_r^{(2)}, \dots, c_r^{(m)} (r \in \mathbb{Z})]$$

is defined as follows:

- for $m = 1$, we set $\text{Pf}[c_r^{(1)}] = c_r^{(1)}$.
- for $m = 2$, we set $\text{Pf}[c_{r_1}^{(1)} c_{r_2}^{(2)}] = c_{r_1}^{(1)} c_{r_2}^{(2)} + 2 \sum_{s=1}^{r_2} (-1)^s c_{r_1+s}^{(1)} c_{r_2-s}^{(2)}$.
- for any even $m \geq 4$, we set

$$\text{Pf}[c_{r_1}^{(1)} \cdots c_{r_m}^{(m)}] = \sum_{s=2}^m (-1)^s \text{Pf}[c_{r_1}^{(1)} c_{r_s}^{(s)}] \cdot \text{Pf}[c_{r_2}^{(2)} \cdots \widehat{c_{r_s}^{(s)}} \cdots c_{r_m}^{(m)}].$$

- for any odd $m \geq 3$, we set

$$\text{Pf}[c_{r_1}^{(1)} \cdots c_{r_m}^{(m)}] = \sum_{s=1}^m (-1)^{s-1} c_{r_s}^{(s)} \cdot \text{Pf}[c_{r_1}^{(1)} \cdots \widehat{c_{r_s}^{(s)}} \cdots c_{r_m}^{(m)}].$$

Remark 4.1 (Pfaffian in classical literature). *If we assume $r_i \geq 0$, $c_0^{(i)} = 1$ and $\text{Pf}[c_{r_i}^{(i)} c_{r_j}^{(j)}] + \text{Pf}[c_{r_j}^{(j)} c_{r_i}^{(i)}] = 0$ hold for all $1 \leq i, j \leq m$, then $\text{Pf}[c_{r_1}^{(1)} \cdots c_{r_m}^{(m)}]$ coincides with Kazarian's Pfaffian. In particular, in the case when m is even, then $\text{Pf}[c_{r_1}^{(1)} \cdots c_{r_m}^{(m)}]$ is the classical Pfaffian of the skew symmetric matrix whose (i, j) entry is given by $\text{Pf}[c_{r_i}^{(i)} c_{r_j}^{(j)}]$. If we further assume that $c_r^{(i)} = c_r^{(j)}$ for all i, j , then it is due to Schur.*

Remark 4.2. *Let λ be a strict partition of length $\ell(\lambda)$, then*

$$\text{Pf}[{}_0\vartheta_{\lambda_1}^{(\lambda_1-1)} \cdots {}_0\vartheta_{\lambda_{\ell(\lambda)}}^{(\lambda_{\ell(\lambda)}-1)}] := \text{Pf}[c_{\lambda_1}^{(1)} \cdots c_{\lambda_{\ell(\lambda)}}^{(\ell(\lambda))}] \Big|_{c_m^{(i)} = {}_0\vartheta_m^{(\lambda_i-1)}} \quad (4.1)$$

is equal to the factorial Q -function $Q_\lambda(x|t)$ defined by Ivanov [13]. This expression is obtained in [11, §11], which is also equivalent to Kazarian's Lagrangian degeneracy loci formula [15]. Note that (4.1) is a variant of Ivanov's original Pfaffian formula [13, Thm 9.1]. In particular, $\text{Pf}[Q_{\lambda_1} \cdots Q_{\lambda_{\ell(\lambda)}}]$ is the classical Schur Q -function [22] where ${}_0\vartheta_r^{(0)}$ is denoted by Q_r in (2.2).

Remark 4.3. *The multi Schur-Pfaffian can be defined in terms of the raising operators (cf. [4]). This aspect will be postponed until Section 7, since we will not use it in the proof of our main theorem.*

Kazarian stated the following properties of Pf in [15, §1]. They follow from the above definition of Pfaffian by the induction on m .

Proposition 4.4.

- (1) *If $\text{Pf}[c_r^{(l)} c_r^{(l)}] = 0$, then we have $\text{Pf}[c_{r_1}^{(l_1)} \cdots c_r^{(l)} c_r^{(l)} \cdots c_{r_m}^{(l_m)}] = 0$.*
- (2) *If $\text{Pf}[c_r^{(l)} c_s^{(l)}] + \text{Pf}[c_s^{(l)} c_r^{(l)}] = 0$, then we have*

$$\text{Pf}[c_{r_1}^{(l_1)} \cdots c_r^{(l)} c_s^{(l)} \cdots c_{r_m}^{(l_m)}] + \text{Pf}[c_{r_1}^{(l_1)} \cdots c_s^{(l)} c_r^{(l)} \cdots c_{r_m}^{(l_m)}] = 0.$$

5. DOUBLE THETA POLYNOMIALS

In this section, first we list basic formulas for the double theta polynomials. In particular, Proposition 5.10 is essential for computing the double Schubert polynomials via the divided difference operators in Section 6. In Section 5.3, we give the geometric interpretation of those polynomials in terms of the Chern classes of vector bundles, although we will not use these facts in the proof of the main theorem.

5.1. ${}_k\vartheta_r^{(l)}(x, z|t)$. Recall Definition 1.1 of the double theta polynomial ${}_k\vartheta_r^{(l)}(x, z|t)$. We denote the generating function by

$${}_k f_l(u) = \sum_{r=0}^{\infty} {}_k\vartheta_r^{(l)}(x, z|t) \cdot u^r. \quad (5.1)$$

Proposition 5.1. *We have ${}_k\vartheta_r^{(l)}(x, z|t) \in \mathcal{R}_{\infty}^{(k)}$.*

Proof. We check the invariance of ${}_k\vartheta_r^{(l)}(x, z | t)$ under the action of s_i^z ($i \geq 0$, $i \neq k$). Since ${}_k\vartheta_r^{(l)}$ is a symmetric polynomial in z_1, \dots, z_k , it is obvious when $i \geq 1$, $i \neq k$. To show $s_0^z({}_k\vartheta_r^{(l)}(x, z | t)) = {}_k\vartheta_r^{(l)}(x, z | t)$, it suffices to consider the case $l = 0$, since s_0^z is $\mathbb{Z}[t]$ -linear. The action of s_0^z is given by substitutions $(x_1, x_2, \dots) \mapsto (z_1, x_1, x_2, \dots)$ and $z_1 \mapsto -z_1$. Thus $s_0^z({}_kf_0(u))$ is

$$\frac{1 + z_1 u}{1 - z_1 u} \prod_{i=1}^{\infty} \frac{1 + x_i u}{1 - x_i u} \cdot (1 - z_1 u) \prod_{i=2}^k (1 + z_i u).$$

Clearly this is equal to ${}_kf_0(u)$. □

We fix $k \geq 0$ throughout this section and denote ${}_k\vartheta_r^{(l)}(x, z | t)$ by $\vartheta_r^{(l)}$ and ${}_k f_l(u)$ by $f_l(u)$.

Lemma 5.2. *Suppose $l \geq 0$. Then we have*

$$\vartheta_r^{(l)} \cdot \vartheta_r^{(l)} + 2 \sum_{j=1}^r (-1)^j \vartheta_{r+j}^{(l)} \cdot \vartheta_{r-j}^{(l)} = \begin{cases} \sum_{s=0}^r e_s(z_1^2, \dots, z_k^2) e_{r-s}(t_1^2, \dots, t_l^2) & \text{if } r \leq k + l, \\ 0 & \text{if } r > k + l. \end{cases}$$

Proof. The claims follow from

$$\begin{aligned} & \prod_{j=1}^k (1 - z_j^2 u^2) \prod_{s=1}^l (1 - t_s^2 u^2) \quad \text{an even degree polynomial in } u \text{ of degree } \leq 2(k + l) \\ &= f_l(u) \cdot f_l(-u) \\ &= \sum_{r \geq 0} \left(\sum_{s=0}^{2r+1} (-1)^s \vartheta_{2r+1-s}^{(l)} \cdot \vartheta_s^{(l)} \right) u^{2r+1} + \sum_{r \geq 0} (-1)^r \left(\vartheta_r^{(l)} \cdot \vartheta_r^{(l)} + 2 \sum_{j=1}^r (-1)^j \vartheta_{r+j}^{(l)} \cdot \vartheta_{r-j}^{(l)} \right) u^{2r}. \end{aligned}$$

□

Lemma 5.3. *For all $l > 1$, we have*

$$\vartheta_r^{(l)} = \vartheta_r^{(l-1)} - t_l \cdot \vartheta_{r-1}^{(l-1)}.$$

For $l \geq 0$, we have

$$\vartheta_r^{(-l)} = \vartheta_r^{(-l-1)} + t_{l+1} \cdot \vartheta_{r-1}^{(-l-1)} = \sum_{i=0}^r (-t_l)^i \vartheta_{r-i}^{(-l+1)}. \quad (5.2)$$

Proof. The first equation is obtained by extracting the coefficient of u^r in the equation

$$f_l(u) = f_{l-1}(u) \cdot (1 - t_l u)$$

which is obvious from the definition of $\vartheta_r^{(l)}$. The second identities are the consequence of the following equations

$$f_{-l}(u) = f_{-l-1}(u) \cdot (1 + t_{l+1} u) = f_{-l+1}(u) (1 + t_{l+1} u)^{-1}.$$

□

Lemma 5.4. *We have*

$$s_i^t(\vartheta_r^{(l)}) = \vartheta_r^{(l)} \quad (l \neq \pm i), \quad (5.3)$$

$$s_i^t(\vartheta_r^{(i)}) = \vartheta_r^{(i-1)} - t_{i+1} \cdot \vartheta_{r-1}^{(i-1)} \quad (i \geq 0), \quad (5.4)$$

$$s_i^t(\vartheta_r^{(-i)}) = \vartheta_r^{(-i-1)} + t_i \cdot \vartheta_{r-1}^{(-i-1)} \quad (i > 0). \quad (5.5)$$

Proof. Since $\vartheta_r^{(l)}$ is a polynomial symmetric in $t_1, \dots, t_{|l|}$, the identity (5.3) for $i \geq 1$ is obvious. The case when $i = 0$, i.e., the invariance of $\vartheta_r^{(l)}$ ($l \neq 0$) under s_0^t , can be shown in the same manner as in the proof of Proposition 5.1.

For $i \geq 1$, we have

$$s_i^t(f_i(u)) = f_0(u)(1 - t_1u) \cdots (1 - t_{i-1}u)(1 - t_{i+1}u) = f_{i-1}(u)(1 - t_{i+1}u).$$

Thus the equation (5.4) for $i \geq 1$ is obtained by comparing the coefficients of u^r . The case when $i = 0$ is derived from the following equation

$$s_0^t(f_0(u)) = \frac{1 - t_1u}{1 + t_1u} \cdot f_0(u) = (1 - t_1u) \cdot f_{-1}(u).$$

The equation (5.5) follows from

$$\begin{aligned} s_i^t(f_{-i}(u)) &= f_0(u)(1 + t_1u)^{-1} \cdots (1 + t_{i-1}u)^{-1}(1 + t_{i+1}u)^{-1} \\ &= f_0(u)(1 + t_1u)^{-1} \cdots (1 + t_{i-1}u)^{-1}(1 + t_iu)^{-1}(1 + t_{i+1}u)^{-1}(1 + t_iu) \\ &= f_{-i-1}(u)(1 + t_iu). \end{aligned}$$

□

Lemma 5.5. *For all $i \geq 0$, we have*

$$\delta_i \vartheta_r^{(l)} = \begin{cases} 0 & \text{if } l \neq \pm i, \\ \vartheta_{r-1}^{(l-1)} & \text{if } l = \pm i. \end{cases}$$

Proof. The case when $l \neq \pm i$ is obvious from the invariance result (5.3). Let $i \geq 1$. The cases when $l = \pm i$ follow from the following equations:

$$\begin{aligned} f_i(u) - s_i^t(f_i(u)) &= f_{i-1}(u)((1 - t_iu) - (1 - t_{i+1}u)) \\ &= f_{i-1}(u)(t_{i+1} - t_i)u, \\ f_{-i}(u) - s_i^t(f_{-i}(u)) &= f_{-i+1}(u)((1 + t_iu)^{-1} - (1 + t_{i+1}u)^{-1}) \\ &= f_{-i+1}(u)(1 + t_iu)^{-1}(1 + t_{i+1}u)^{-1}(t_{i+1} - t_i) \\ &= f_{-i-1}(u)(t_{i+1} - t_i)u. \end{aligned}$$

Finally we show $\delta_0(\vartheta_r^{(0)}) = \vartheta_{r-1}^{(-1)}$. This is a consequence of the following equation:

$$f_0(u) - s_0^t(f_0(u)) = f_0(u) \left(1 - \frac{1 - t_1u}{1 + t_1u} \right) = f_0(u) \frac{2t_1u}{1 + t_1u} = f_{-1}(u)2t_1u.$$

□

Lemma 5.6. *For $i > 0$, we have*

$$\delta_i(\vartheta_r^{(i)} \cdot \vartheta_s^{(-i)}) = \vartheta_{r-1}^{(i-1)} \cdot \vartheta_s^{(-i-1)} + \vartheta_r^{(i-1)} \cdot \vartheta_{s-1}^{(-i-1)}.$$

Proof. By the Leibnitz rule, Lemma 5.5, Equation (5.4), and Equation (5.2), we have

$$\begin{aligned} \delta_i(\vartheta_r^{(i)} \cdot \vartheta_s^{(-i)}) &= \delta_i(\vartheta_r^{(i)})\vartheta_s^{(-i)} + s_i(\vartheta_r^{(i)})\delta_i(\vartheta_s^{(-i)}) \\ &= \vartheta_{r-1}^{(i-1)} \cdot \vartheta_s^{(-i)} + (\vartheta_r^{(i-1)} - t_{i+1} \cdot \vartheta_{r-1}^{(i-1)})\vartheta_{s-1}^{(-i-1)} \\ &= \vartheta_{r-1}^{(i-1)}(\vartheta_s^{(-i)} - t_{i+1} \cdot \vartheta_{s-1}^{(-i-1)}) + \vartheta_r^{(i-1)} \cdot \vartheta_{s-1}^{(-i-1)} \\ &= \vartheta_{r-1}^{(i-1)} \cdot \vartheta_s^{(-i-1)} + \vartheta_r^{(i-1)} \cdot \vartheta_{s-1}^{(-i-1)}. \end{aligned}$$

□

We use the following notation in the rest of the paper.

Definition 5.7. For all $(r_1, \dots, r_m), (l_1, \dots, l_m) \in \mathbb{Z}^m$, let

$$\text{Pf} [\vartheta_{r_1}^{(l_1)} \vartheta_{r_2}^{(l_2)} \dots \vartheta_{r_m}^{(l_m)}] := \text{Pf} [c_{r_1}^{(1)} c_{r_2}^{(2)} \dots c_{r_m}^{(m)}] \Big|_{c=\vartheta^{(l)}},$$

where $|_{c=\vartheta^{(l)}}$ means that we substitute $\vartheta_s^{(l_i)}$ to $c_s^{(i)}$ for all $i \in \{1, \dots, m\}$ and $s \in \mathbb{Z}$.

We emphasize that we substitute theta polynomials after we write the Pfaffian as polynomials in the formal variable $c_s^{(i)}$'s. For example,

$$\text{Pf} [\vartheta_{-1}^{(l_1)} \vartheta_1^{(l_2)}] = \text{Pf} [c_{-1}^{(1)} c_1^{(2)}] \Big|_{c=\vartheta^{(l)}} = (c_{-1}^{(1)} c_1^{(2)} - 2c_0^{(1)} c_0^{(2)}) \Big|_{c=\vartheta^{(l)}} = -2\vartheta_0^{(l_1)} \vartheta_0^{(l_2)} = -2.$$

Remark 5.8. The following are clear from the definition of Pfaffian and the facts that $\vartheta_r^{(l)} = 0$ for all $r < 0$ and $\vartheta_0^{(l)} = 1$.

$$\begin{aligned} \text{Pf} [\vartheta_{r_1}^{(l_1)} \dots \vartheta_{r_m}^{(l_m)} \vartheta_0^{(l_{m+1})}] &= \text{Pf} [\vartheta_{r_1}^{(l_1)} \dots \vartheta_{r_m}^{(l_m)}], \\ \text{Pf} [\vartheta_{r_1}^{(l_1)} \dots \vartheta_{r_m}^{(l_m)} \vartheta_{r_{m+1}}^{(l_{m+1})}] &= 0 \quad \text{if } r_{m+1} < 0. \end{aligned}$$

We use the following two propositions from Lemma 5.2, 5.5, and 5.6 and use them in the proofs of our main results.

Proposition 5.9. Suppose $l \geq 0$ and $r > k + l$. Then

$$\text{Pf} [\vartheta_{r_1}^{(l_1)} \dots \vartheta_r^{(l)} \vartheta_r^{(l)} \dots \vartheta_{r_m}^{(l_m)}] = 0.$$

Proof. Lemma 5.2 states that $\text{Pf} [\vartheta_r^{(l)} \vartheta_r^{(l)}] = 0$ if $l \geq 0$ and $r > k + l$. Thus the claim follows from Proposition 4.4 (1). □

Proposition 5.10.

- (a) Let $i \geq 0$. If $l_p \neq \pm i$ for all $p \in \{1, \dots, m\}$, then $\delta_i \text{Pf} [\vartheta_{r_1}^{(l_1)} \dots \vartheta_{r_m}^{(l_m)}] = 0$.
- (b) Let $i \geq 0$. Suppose that $l_p \in \{\pm i\}$ for some $p \in \{1, \dots, m\}$ and that $l_q \notin \{\pm i\}$ for all $q \neq p$.

$$\delta_i \text{Pf} [\vartheta_{r_1}^{(l_1)} \dots \vartheta_{r_p}^{(l_p)} \dots \vartheta_{r_m}^{(l_m)}] = \text{Pf} [\vartheta_{r_1}^{(l_1)} \dots \vartheta_{r_p-1}^{(l_p-1)} \dots \vartheta_{r_m}^{(l_m)}].$$

- (c) Let $i > 0$. Suppose that $l_p = i$ and $l_q = -i$ for some $p < q$ and that $l_s \neq \pm i$ for all $s \notin \{p, q\}$. Then we have

$$\begin{aligned} & \delta_i \text{Pf} [\vartheta_{r_1}^{(l_1)} \dots \vartheta_{r_p}^{(i)} \dots \vartheta_{r_q}^{(-i)} \dots \vartheta_{r_m}^{(l_m)}] \\ &= \text{Pf} [\vartheta_{r_1}^{(l_1)} \dots \vartheta_{r_p-1}^{(i-1)} \dots \vartheta_{r_q}^{(-i-1)} \dots \vartheta_{r_m}^{(l_m)}] + \text{Pf} [\vartheta_{r_1}^{(l_1)} \dots \vartheta_{r_p}^{(i-1)} \dots \vartheta_{r_q-1}^{(-i-1)} \dots \vartheta_{r_m}^{(l_m)}]. \end{aligned}$$

Proof. We can prove (a) and (b) from Lemma 5.5 by induction on m in the axioms of Pfaffian. We show how to prove (c). If $m = 2$, the claim follows by applying Lemma 5.6 to each monomial in the definition of Pfaffian. Suppose that m is odd > 2 , and that the claim holds for all $m' < m$. By the definition of Pfaffian, Leibnitz rule, and Lemma 5.4, together with (a) and (b) above, we

can compute

$$\begin{aligned}
& \delta_i \text{Pf}[\vartheta_{r_1}^{(l_1)} \dots \vartheta_{r_m}^{(l_m)}] \\
&= \sum_{s=1}^m (-1)^{s-1} \delta_i \left(\vartheta_{r_s}^{(l_s)} \right) \text{Pf}[\vartheta_{r_1}^{(l_1)} \dots \widehat{\vartheta_{r_s}^{(l_s)}} \dots \vartheta_{r_m}^{(l_m)}] + (-1)^{s-1} s_i \left(\vartheta_{r_s}^{(l_s)} \right) \delta_i \text{Pf}[\vartheta_{r_1}^{(l_1)} \dots \widehat{\vartheta_{r_s}^{(l_s)}} \dots \vartheta_{r_m}^{(l_m)}] \\
&= (-1)^{p-1} \vartheta_{r_p-1}^{(i-1)} \text{Pf}[\vartheta_{r_1}^{(l_1)} \dots \widehat{\vartheta_{r_p}^{(i)}} \dots \vartheta_{r_q}^{(l_q)} \dots \vartheta_{r_m}^{(l_m)}] \\
&\quad + (-1)^{p-1} \left(\vartheta_{r_p}^{(i-1)} - t_{i+1} \vartheta_{r_p-1}^{(i-1)} \right) \text{Pf}[\vartheta_{r_1}^{(l_1)} \dots \widehat{\vartheta_{r_p}^{(i)}} \dots \vartheta_{r_q-1}^{(-i-1)} \dots \vartheta_{r_m}^{(l_m)}] \\
&\quad + (-1)^{q-1} \vartheta_{r_q-1}^{(-i-1)} \text{Pf}[\vartheta_{r_1}^{(l_1)} \dots \vartheta_{r_p}^{(i)} \dots \widehat{\vartheta_{r_q}^{(-i)}} \dots \vartheta_{r_m}^{(l_m)}] \\
&\quad + (-1)^{q-1} \left(\vartheta_{r_q}^{(-i-1)} + t_i \vartheta_{r_q-1}^{(-i-1)} \right) \text{Pf}[\vartheta_{r_1}^{(l_1)} \dots \vartheta_{r_p-1}^{(i-1)} \dots \widehat{\vartheta_{r_q}^{(-i)}} \dots \vartheta_{r_m}^{(l_m)}] \\
&\quad + \sum_{s \in \{1, \dots, m\} \setminus \{p, q\}} (-1)^{s-1} \vartheta_{r_s}^{(l_s)} \text{Pf}[\vartheta_{r_1}^{(l_1)} \dots \widehat{\vartheta_{r_s}^{(l_s)}} \dots \vartheta_{r_p-1}^{(i-1)} \dots \vartheta_{r_q}^{(-i-1)} \dots \vartheta_{r_m}^{(l_m)}] \\
&\quad + \sum_{s \in \{1, \dots, m\} \setminus \{p, q\}} (-1)^{s-1} \vartheta_{r_s}^{(l_s)} \text{Pf}[\vartheta_{r_1}^{(l_1)} \dots \widehat{\vartheta_{r_s}^{(l_s)}} \dots \vartheta_{r_p}^{(i-1)} \dots \vartheta_{r_q-1}^{(-i-1)} \dots \vartheta_{r_m}^{(l_m)}].
\end{aligned}$$

By Lemma 5.3 and the natural multilinearity of Pfaffian, the sum of the first and second terms is

$$(-1)^{p-1} \vartheta_{r_p-1}^{(i-1)} \text{Pf}[\vartheta_{r_1}^{(l_1)} \dots \widehat{\vartheta_{r_p}^{(i)}} \dots \vartheta_{r_q}^{(-i-1)} \dots \vartheta_{r_m}^{(l_m)}] + (-1)^{p-1} \vartheta_{r_p}^{(i-1)} \text{Pf}[\vartheta_{r_1}^{(l_1)} \dots \widehat{\vartheta_{r_p}^{(i)}} \dots \vartheta_{r_q-1}^{(-i-1)} \dots \vartheta_{r_m}^{(l_m)}].$$

Similarly, the sum of the third and fourth terms is

$$(-1)^{q-1} \vartheta_{r_q-1}^{(-i-1)} \text{Pf}[\vartheta_{r_1}^{(l_1)} \dots \vartheta_{r_p}^{(i)} \dots \widehat{\vartheta_{r_q}^{(-i)}} \dots \vartheta_{r_m}^{(l_m)}] + (-1)^{q-1} \vartheta_{r_q}^{(-i-1)} \text{Pf}[\vartheta_{r_1}^{(l_1)} \dots \vartheta_{r_p-1}^{(i-1)} \dots \widehat{\vartheta_{r_q}^{(-i)}} \dots \vartheta_{r_m}^{(l_m)}].$$

Thus by the definition of Pfaffian again, we obtain the desired formula. The case when m is even can be proved similarly. \square

5.2. Supplementary results on ϑ -functions. In this section, we generalize Proposition 5.9 and Proposition 5.10, which hold in \mathcal{R}_∞ . We will use them only in Section 8.

Definition 5.11. For each $(r_1, \dots, r_m), (l_1, \dots, l_m) \in \mathbb{Z}^m$ and $(k_1, \dots, k_m) \in (\mathbb{Z}_{\geq 0})^m$, let

$$\text{Pf} \left[k_1 \vartheta_{r_1}^{(l_1)} k_2 \vartheta_{r_2}^{(l_2)} \dots k_m \vartheta_{r_m}^{(l_m)} \right] := \text{Pf} \left[c_{r_1}^{(1)} c_{r_2}^{(2)} \dots c_{r_m}^{(m)} \right] \Big|_{c=k\vartheta^{(l)}},$$

where $|_{c=k\vartheta^{(l)}}$ means that we substitute $k_i \vartheta_s^{(l_i)}$ to $c_s^{(i)}$ for all $i \in \{1, \dots, m\}$ and $s \in \mathbb{Z}$.

Remark 5.12. By the definition of Pfaffian and the fact that $k_r \vartheta_r^{(l)} = 0$ for all $r < 0$ and $k_0 \vartheta_0^{(l)} = 1$, we have

$$\begin{aligned}
& \text{Pf}[k_1 \vartheta_{r_1}^{(l_1)} \dots k_m \vartheta_{r_m}^{(l_m)} k_{m+1} \vartheta_0^{(l_{m+1})}] = \text{Pf}[k_1 \vartheta_{r_1}^{(1)} \dots k_m \vartheta_{r_m}^{(m)}], \\
& \text{Pf}[k_1 \vartheta_{r_1}^{(l_1)} \dots k_m \vartheta_{r_m}^{(l_m)} k_{m+1} \vartheta_{r_{m+1}}^{(l_{m+1})}] = 0 \quad \text{if } r_{m+1} < 0.
\end{aligned}$$

Now by Lemma 5.2 with the help of Proposition 4.4 (1), we have the following.

Proposition 5.13. Suppose $l \geq 0$ and $r > k + l$. Then

$$\text{Pf}[k_1 \vartheta_{r_1}^{(l_1)} \dots k \vartheta_r^{(l)} k \vartheta_r^{(l)} \dots k_m \vartheta_{r_m}^{(l_m)}] = 0.$$

The proofs of the following lemma and proposition are identical to the ones of Lemma 5.6 and Proposition 5.10.

Lemma 5.14. *For $i > 0$, we have*

$$\delta_i(k_1\vartheta_r^{(i)} \cdot k_2\vartheta_s^{(-i)}) = k_1\vartheta_{r-1}^{(i-1)} \cdot k_2\vartheta_s^{(-i-1)} + k_1\vartheta_r^{(i-1)} \cdot k_2\vartheta_{s-1}^{(-i-1)}.$$

Proposition 5.15. *Let $i \geq 0$.*

(a) *If $l_p \neq \pm i$ for all p , then $\delta_i \text{Pf}[k_1\vartheta_{r_1}^{(l_1)} \cdots k_m\vartheta_{r_m}^{(l_m)}] = 0$.*

(b) *Suppose that $l_p \in \{\pm i\}$ for some p and that $l_q \notin \{\pm i\}$ for all $q \neq p$. Then we have*

$$\delta_i \text{Pf}[k_1\vartheta_{r_1}^{(l_1)} \cdots k_p\vartheta_{r_p}^{(\pm i)} \cdots k_m\vartheta_{r_m}^{(l_m)}] = \text{Pf}[k_1\vartheta_{r_1}^{(l_1)} \cdots k_p\vartheta_{r_p-1}^{(\pm i-1)} \cdots k_m\vartheta_{r_m}^{(l_m)}].$$

(c) *Suppose that $i \neq 0$ and that $l_p = i$ and $l_q = -i$ for some $p < q$ and that $l_s \in \{\pm i\}$ for all $s \notin \{p, q\}$. Then we have*

$$\begin{aligned} & \delta_i \text{Pf}[k_1\vartheta_{r_1}^{(l_1)} \cdots k_p\vartheta_{r_p}^{(i)} \cdots k_q\vartheta_{r_q}^{(-i)} \cdots k_m\vartheta_{r_m}^{(l_m)}] \\ &= \text{Pf}[k_1\vartheta_{r_1}^{(l_1)} \cdots k_p\vartheta_{r_p-1}^{(i-1)} \cdots k_q\vartheta_{r_q}^{(-i-1)} \cdots k_m\vartheta_{r_m}^{(l_m)}] + \text{Pf}[k_1\vartheta_{r_1}^{(l_1)} \cdots k_p\vartheta_{r_p}^{(i-1)} \cdots k_q\vartheta_{r_q-1}^{(-i-1)} \cdots k_m\vartheta_{r_m}^{(l_m)}]. \end{aligned}$$

5.3. Double theta polynomials as equivariant Chern classes. In this section, we show that the double theta polynomials ${}_k\vartheta_r^{(l)}$ correspond to the Chern classes of vector bundles. The result is not used in the proof of the main theorem.

Let \mathcal{E} be the trivial vector bundle of rank $2n$ over $\mathcal{F}l_n$, and $\mathcal{L}_i, \mathcal{L}_i^* \subset \mathcal{E}$ the subbundles whose fibers are $\text{Span}(\mathbf{e}_i), \text{Span}(\mathbf{e}_i^*)$ respectively. Let $T = (\mathbb{C}^\times)^n$ be the n -dimensional torus and let t_1, \dots, t_n be the standard basis of $\mathfrak{t}_{\mathbb{Z}}^*$. Then T acts on \mathcal{L}_i with the weight $-t_i$ and \mathcal{L}_i^* with the weight t_i . Note that $t_i = -c_1^T(\mathcal{L}_i)$. Let

$$\mathcal{L} = \bigoplus_{i=1}^n \mathcal{L}_i, \quad \mathcal{L}^* = \bigoplus_{i=1}^n \mathcal{L}_i^*,$$

and hence $\mathcal{E} = \mathcal{L} \oplus \mathcal{L}^*$. Let $\mathcal{V}_n \subset \cdots \subset \mathcal{V}_1 = \mathcal{V} \subset \mathcal{E}$ be the tautological flag of vector bundles over the complete flag variety $\mathcal{F}l_n$ of isotropic subspaces of V where $\text{rank } \mathcal{V}_i = n - i + 1$. Let $\mathbf{z}_i = c_1^T(\mathcal{V}_i/\mathcal{V}_{i+1})$. Note that \mathcal{V}_{k+1} is the pullback of the tautological subbundle of rank $n - k$ on SG_n^k along the natural projection $pr_k : \mathcal{F}l_n \rightarrow SG_n^k$. Note that $\mathcal{Q} := \mathcal{E}/\mathcal{V}_{k+1}$ is the universal quotient bundle of SG_n^k .

From the geometric construction of $\pi_n : \mathcal{R}_\infty \rightarrow H_T^*(\mathcal{F}l_n)$ ([11, §10]), we have $\pi_n(Q_r(x)) = c_r(\mathcal{L}^* - \mathcal{V}_1) = c_r(\mathcal{V}_1^* - \mathcal{L})$. In other words,

$$\pi_n : \prod_{i=0}^{\infty} \frac{1+x_i u}{1-x_i u} \mapsto \prod_{i=1}^n \frac{1+t_i u}{1+\mathbf{z}_i u} = \prod_{i=1}^n \frac{1-\mathbf{z}_i u}{1-t_i u}. \quad (5.6)$$

Define

$$\mathcal{U}_l := \bigoplus_{i=l}^n \mathcal{L}_i \quad \text{if } l > 0, \quad \mathcal{U}_{-l} := \mathcal{L} \oplus \bigoplus_{i=1}^{l+1} \mathcal{L}_i^* \quad \text{if } l \geq 0.$$

Proposition 5.16. *For all $-n \leq l \leq n-1$, we have*

$$\pi_n \left({}_k\vartheta_r^{(l)}(x, z | t) \right) = c_r^T(\mathcal{E} - \mathcal{V}_{k+1} - \mathcal{U}_{l+1}).$$

In particular, we have

$$c_r^T(\mathcal{E} - \mathcal{V}_{k+1} - \mathcal{U}_{r-k}) = [\Omega_r]_T \quad (1 \leq r \leq n+k).$$

Proof. In view of the relation (5.6), the proposition can be shown by the following formal calculations. For $l \geq 0$, we have

$$c^T(\mathcal{E} - \mathcal{V}_{k+1} - \mathcal{U}_{l+1}) = \frac{\prod_{i=1}^n (1 - t_i^2 u^2)}{\prod_{i=k+1}^n (1 + \mathbf{z}_i u) \prod_{i=l+1}^n (1 - t_i u)} = \prod_{i=1}^n \frac{1 + t_i u}{1 + \mathbf{z}_i u} \prod_{i=1}^k (1 + \mathbf{z}_i u) \prod_{i=1}^l (1 - t_i u),$$

and for $l > 0$, we have

$$\begin{aligned} c^T(\mathcal{E} - \mathcal{V}_{k+1} - \mathcal{U}_{-l+1}) &= \frac{\prod_{i=1}^n (1 - t_i^2 u^2)}{\prod_{i=k+1}^n (1 + \mathbf{z}_i u) \prod_{i=1}^n (1 - t_i u) \prod_{i=1}^l (1 + t_i u)} \\ &= \prod_{i=1}^n \frac{1 + t_i u}{1 + \mathbf{z}_i u} \prod_{i=1}^k (1 + \mathbf{z}_i u) \prod_{i=1}^l \frac{1}{1 + t_i u}. \end{aligned}$$

The second statement follows from the result (1.3) due to Wilson. \square

6. PROOF OF THE MAIN THEOREM

Fix $k \geq 0$. We omit k in this section and the next, *i.e.* we denote $\vartheta_r^{(l)} = {}_k\vartheta_r^{(l)}$.

Recall that for $w \in W_n^{(k)}$ we defined $w^\vee \in W_n^{(k)}$ in §3.4.

Definition 6.1. Let

$$\Theta_{\max}^{(n,k)}(x, z | t) := \text{Pf}[\vartheta_{n+k}^{(n-1)} \vartheta_{n+k-1}^{(n-2)} \cdots \vartheta_{2k+1}^{(k)}].$$

For each $\lambda \in \mathcal{P}_n^{(k)}$, define

$$\Theta_\lambda^{(n,k)}(x, z | t) := \delta_{(w_\lambda^{(k)})^\vee} \Theta_{\max}^{(n,k)}(x, z | t). \quad (6.1)$$

Theorem 6.2 (Pfaffian sum formula for Θ_λ). *Let $\lambda \in \mathcal{P}_n^{(k)}$. We have*

$$\Theta_\lambda^{(n,k)}(x, z | t) = \sum_{I \subset D(\lambda)} \text{Pf} \left[\vartheta_{\lambda_1 + a_1^I}^{(\chi_1)} \cdots \vartheta_{\lambda_{n-k} + a_{n-k}^I}^{(\chi_{n-k})} \right], \quad (6.2)$$

where I runs over all subsets of $D(\lambda)$ and $a_s^I = \#\{j \mid (s, j) \in I\} - \#\{i \mid (i, s) \in I\}$.

Proof. For simplicity of notation, we drop the superscript (k) from $w_\lambda^{(k)}$. We proceed by induction on $\ell(w_\lambda^\vee)$. If $\ell(w_\lambda^\vee) = 0$, then $w_\lambda = w_{\max}$, and the result is obvious from the definition. Suppose that $\ell(w_\lambda^\vee) > 0$. There is a strict k -partition λ' and $i \in \{0, 1, \dots, n-1\}$ such that $w_{\lambda'} \in W_n^{(k)}$, $s_i w_{\lambda'} = w_\lambda$, and $\ell(w_\lambda) = \ell(w_{\lambda'}) - 1$. By Lemma 3.4 and 3.5, $w_{\lambda'}$ is in one of the cases $L1, L2, L3$ and $L0$. Let $\chi_\lambda = (\chi_1, \dots, \chi_{n-k})$ and $\chi_{\lambda'} = (\chi'_1, \dots, \chi'_{n-k})$ be the characteristic indices of λ and λ' respectively. By the induction hypothesis we have

$$\delta_{w_{\lambda'}^\vee} \text{Pf}[\vartheta_{n+k}^{(n-1)} \vartheta_{n+k-1}^{(n-2)} \cdots \vartheta_{2k+1}^{(k)}] = \sum_{I \subset D(\lambda')} \text{Pf} \left[\vartheta_{\lambda'_1 + a_1^I}^{(\chi'_1)} \cdots \vartheta_{\lambda'_{n-k} + a_{n-k}^I}^{(\chi'_{n-k})} \right].$$

In the cases $L2$, $L3$, or $L0$, we have $D(\lambda') = D(\lambda)$. Furthermore, for some p , $\chi'_p = \pm i = \chi_p + 1$ and $\lambda'_p = \lambda_p + 1$; $\chi'_q = \chi_q$ and $\lambda_q = \lambda'_q$ for all $q \neq p$. Thus by Proposition 5.10, we can compute

$$\begin{aligned}
& \delta_{w_\lambda^\vee} \text{Pf}[\vartheta_{n+k}^{(n-1)} \vartheta_{n+k-1}^{(n-2)} \cdots \vartheta_{2k+1}^{(k)}] \\
&= \delta_i \delta_{w_{\lambda'}^\vee} \text{Pf}[\vartheta_{n+k}^{(n-1)} \vartheta_{n+k-1}^{(n-2)} \cdots \vartheta_{2k+1}^{(k)}] \\
&= \sum_{I \subset D(\lambda')} \delta_i \text{Pf} \left[\vartheta_{\lambda'_1 + a_1^I}^{(\chi'_1)} \cdots \vartheta_{\lambda'_a + a_p^I}^{(\pm i)} \cdots \vartheta_{\lambda'_{n-k} + a_{n-k}^I}^{(\chi'_{n-k})} \right] \\
&= \sum_{I \subset D(\lambda)} \text{Pf} \left[\vartheta_{\lambda'_1 + a_1^I}^{(\chi'_1)} \cdots \vartheta_{\lambda'_p - 1 + a_p^I}^{(\pm i - 1)} \cdots \vartheta_{\lambda'_{n-k} + a_{n-k}^I}^{(\chi'_{n-k})} \right] \\
&= \sum_{I \subset D(\lambda)} \text{Pf} \left[\vartheta_{\lambda_1 + a_1^I}^{(\chi_1)} \cdots \vartheta_{\lambda_p + a_p^I}^{(\chi_p)} \cdots \vartheta_{\lambda_{n-k} + a_{n-k}^I}^{(\chi_{n-k})} \right],
\end{aligned}$$

where the first equality follows from Proposition 3.18, the second is the induction hypothesis, and the third follows by Proposition 5.10.

In the case $L1$, we have $D(\lambda) = D(\lambda') \sqcup \{(p, q)\}$. Furthermore $\chi'_p = i = \chi_p + 1$, $\chi'_q = -i = \chi_q + 1$, $\lambda_p = \lambda'_p - 1$ and $\lambda'_q = \lambda_q$ for some p and q ; $\chi'_r = \chi_r$ and $\lambda_r = \lambda'_r$ for all $r \neq p, q$. Here note that $\lambda'_p \neq 0$ so that $\lambda_p \geq 0$. The claim now follows from the computation:

$$\begin{aligned}
& \delta_{w_\lambda^\vee} \text{Pf}[\vartheta_{n+k}^{(n-1)} \vartheta_{n+k-1}^{(n-2)} \cdots \vartheta_{2k+1}^{(k)}] \\
&= \delta_i \delta_{w_{\lambda'}^\vee} \text{Pf}[\vartheta_{n+k}^{(n-1)} \vartheta_{n+k-1}^{(n-2)} \cdots \vartheta_{2k+1}^{(k)}] \\
&= \sum_{I \subset D(\lambda')} \delta_i \text{Pf} \left[\vartheta_{\lambda'_1 + a_1^I}^{(\chi'_1)} \cdots \vartheta_{\lambda'_p + a_p^I}^{(i)} \cdots \vartheta_{\lambda'_q + a_q^I}^{(-i)} \cdots \vartheta_{\lambda'_{n-k} + a_{n-k}^I}^{(\chi'_{n-k})} \right] \\
&= \sum_{I \subset D(\lambda')} \left(\text{Pf} \left[\vartheta_{\lambda'_1 + a_1^I}^{(\chi'_1)} \cdots \vartheta_{\lambda'_p + a_p^I - 1}^{(i-1)} \cdots \vartheta_{\lambda'_q + a_q^I}^{(-i-1)} \cdots \vartheta_{\lambda'_{n-k} + a_{n-k}^I}^{(\chi'_{n-k})} \right] \right. \\
&\quad \left. + \text{Pf} \left[\vartheta_{\lambda'_1 + a_1^I}^{(\chi'_1)} \cdots \vartheta_{\lambda'_p + a_p^I}^{(i-1)} \cdots \vartheta_{\lambda'_q + a_q^I - 1}^{(-i-1)} \cdots \vartheta_{\lambda'_{n-k} + a_{n-k}^I}^{(\chi'_{n-k})} \right] \right) \\
&= \sum_{I \subset D(\lambda')} \text{Pf} \left[\vartheta_{\lambda_1 + a_1^I}^{(\chi_1)} \cdots \vartheta_{\lambda_p + a_p^I}^{(\chi_p)} \cdots \vartheta_{\lambda_q + a_q^I}^{(\chi_q)} \cdots \vartheta_{\lambda_{n-k} + a_{n-k}^I}^{(\chi_{n-k})} \right] \\
&\quad + \sum_{I \subset D(\lambda')} \text{Pf} \left[\vartheta_{\lambda_1 + a_1^I}^{(\chi_1)} \cdots \vartheta_{\lambda_p + a_p^I + 1}^{(\chi_p)} \cdots \vartheta_{\lambda_q + a_q^I - 1}^{(\chi_q)} \cdots \vartheta_{\lambda_{n-k} + a_{n-k}^I}^{(\chi_{n-k})} \right] \\
&= \sum_{I \subset D(\lambda')} \text{Pf} \left[\vartheta_{\lambda_1 + a_1^I}^{(\chi_1)} \cdots \vartheta_{\lambda_p + a_p^I}^{(\chi_p)} \cdots \vartheta_{\lambda_q + a_q^I}^{(\chi_q)} \cdots \vartheta_{\lambda_{n-k} + a_{n-k}^I}^{(\chi_{n-k})} \right] \\
&\quad + \sum_{(p,q) \in I \subset D(\lambda)} \text{Pf} \left[\vartheta_{\lambda_1 + a_1^I}^{(\chi_1)} \cdots \vartheta_{\lambda_p + a_p^I}^{(\chi_p)} \cdots \vartheta_{\lambda_q + a_q^I}^{(\chi_q)} \cdots \vartheta_{\lambda_{n-k} + a_{n-k}^I}^{(\chi_{n-k})} \right] \\
&= \sum_{I \subset D(\lambda)} \text{Pf} \left[\vartheta_{\lambda_1 + a_1^I}^{(\chi_1)} \cdots \vartheta_{\lambda_p + a_p^I}^{(\chi_p)} \cdots \vartheta_{\lambda_q + a_q^I}^{(\chi_q)} \cdots \vartheta_{\lambda_{n-k} + a_{n-k}^I}^{(\chi_{n-k})} \right],
\end{aligned}$$

where the first equality follows from Proposition 3.18, the second is the induction hypothesis, the third follows by Proposition 5.10, and the second last equality holds, since, for each $I \in D(\lambda')$

and $J := I \cup \{(p, q)\} \in D(\lambda)$, a^I and a^J are related by

$$a_p^I + 1 = a_p^J, \quad a_q^I - 1 = a_q^J, \quad \text{and} \quad a_r^I = a_r^J \quad \forall r \neq p, q.$$

□

Remark 6.3. We owe H. Naruse for pointing out, in the early stage of this work, that $\mathfrak{C}_{w_{max}}$ has a Pfaffian expression.

Remark 6.4. The expression of the right hand side of Theorem 6.2 is essentially independent of n . More precisely, if $\lambda \in \mathcal{P}_n^{(k)}$, then we have obviously $\lambda \in \mathcal{P}_{n+1}^{(k)}$ and all nonzero Pfaffians appearing in the formulas for $\Theta_\lambda^{(n,k)}$ and $\Theta_\lambda^{(n+1,k)}$ naturally coincide. We have $\Theta_\lambda^{(n,k)} = \Theta_\lambda^{(n+1,k)}$, in particular. This fact can be checked by using Remark 5.8. In fact, one can check that the lower indexes (degree) of the right end ϑ in the Pfaffians appearing in the formula of $\Theta_\lambda^{(n+1,k)}$ are less than or equal to zero.

Proposition 6.5 (Stability of Θ_λ). *Let $\lambda \in \mathcal{P}_n^{(k)}$. For all $m \geq n$, we have*

$$\Theta_\lambda^{(m,k)}(x, z | t) = \Theta_\lambda^{(n,k)}(x, z | t).$$

Proof. This is a consequence of the Pfaffian sum formula in Theorem 6.2 (see Remark 6.4). □

By the above proposition and Proposition 5.1, for each $\lambda \in \mathcal{P}_\infty^{(k)}$, we can define $\Theta_\lambda^{(k)}(x, z | t)$ to be the element of $\mathcal{R}_\infty^{(k)}$ such that $\Theta_\lambda^{(k)}(x, z | t) = \Theta_\lambda^{(n,k)}(x, z | t)$ for any n such that $\lambda \in \mathcal{P}_n^{(k)}$.

Lemma 6.6. *We have $\delta_j \Theta_{max}^{(n,k)} = 0$ for $j \neq k$.*

Proof. If $j \neq k$, then we are in the situation of (a) or (b) in Proposition 5.10. If (b) is the case, then the claim follows from Proposition 4.4 (together with Lemma 5.2). □

Proposition 6.7. *Let $\lambda \in \mathcal{P}_\infty^{(k)}$ and $w_\lambda^{(k)}$ the corresponding element in $W_\infty^{(k)}$. We have*

$$\delta_i \Theta_\lambda^{(k)} = \begin{cases} \Theta_\mu^{(k)} & \text{if } s_i w_\lambda^{(k)} = w_\mu^{(k)} \ (\mu \in \mathcal{P}_\infty^{(k)}, \mu \subset \lambda, |\mu| = |\lambda| - 1), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By the definition of $\Theta_\lambda^{(n,k)}$ and the fact $\delta_i \Theta_{max}^{(n,k)} = 0$ for $i \neq k$ (Lemma 6.6), the result follows from Proposition 3.18 immediately. □

Lemma 6.8. *We have*

$$\Theta_\lambda^{(k)}|_\emptyset = \delta_{\lambda, \emptyset},$$

where the notation $|_\mu$ is defined in Definition 3.8, and \emptyset denotes the empty partition.

Proof. We have $\Theta_\emptyset^{(k)} = 1$ since ${}_k \vartheta_0^{(\ell)} = 1$ and ${}_k \vartheta_m^{(\ell)} = 0$ for all $m < 0$, and hence $\Theta_\emptyset^{(k)}|_\emptyset = 1$.

Now assume $\lambda \neq \emptyset$. For each $g \in \mathcal{R}_\infty^{(k)}$, the polynomial $g|_\emptyset \in \mathbb{Z}[t]$ is given by the specializations

$$(z_1, \dots, z_k) \mapsto (t_1, \dots, t_k) \quad \text{and} \quad (x_1, x_2, \dots) \mapsto (0, 0, \dots).$$

The generating function of ϑ -functions in Definition 1.1 becomes a polynomial in u of degree $k+l$ after we specialize it as above. Thus, by the degree reason, we have

$${}_k \vartheta_m^{(\ell)}|_\emptyset = 0 \quad \text{if } \ell + k < m \text{ and } 0 < m. \quad (6.3)$$

We can expand the right hand side of (6.2) as a polynomial in terms of the ϑ -functions ${}_k\vartheta_m^{(l)}$ by using the definition of Pf in Section 4, in such a way that each monomial contains ${}_k\vartheta_{\lambda_1+a_1^I+j}^{(\chi_1)}$ (the first factor) for some $j \geq 0$. Note that $a_1^I \geq 0$ since $\{i \mid (i, 1) \in I\}$ is always empty. Since $0 < \lambda_1 \leq \lambda_1 + a_1^I + j$ and $\chi_1 + k = \lambda_1 - 1 < \lambda_1$ (see (3.1)), we have ${}_k\vartheta_{\lambda_1+a_1^I+j}^{(\chi_1)} \Big|_{\emptyset} = 0$ by (6.3). Therefore $\Theta_\lambda^{(k)} \Big|_{\emptyset} = 0$ for all $\lambda \in \mathcal{P}_\infty^{(k)}$. \square

Theorem 6.9. *Let $\lambda \in \mathcal{P}_\infty^{(k)}$. We have*

$$\mathfrak{C}_{w_\lambda^{(k)}}(z, t; x) = \Theta_\lambda^{(k)}(x, z \mid t).$$

Proof. By Proposition 5.1 and Proposition 6.2, we have $\Theta_\lambda^{(k)} \in \mathcal{R}_\infty^{(k)}$. By Proposition 6.7 and Lemma 6.8, the family $\Theta_\lambda^{(k)}, \lambda \in \mathcal{P}_\infty^{(k)}$ satisfies the equations (3.8), (3.9) in Lemma 3.11. Therefore, by the uniqueness, $\mathfrak{C}_{w_\lambda^{(k)}}$ must coincide with $\Theta_\lambda^{(k)}$ for all $\lambda \in \mathcal{P}_\infty^{(k)}$. \square

Corollary 6.10. *The ring $\mathcal{R}_\infty^{(k)}$ is generated by ${}_k\vartheta_r^{(0)}$ ($r \geq 1$) as an algebra over $\mathbb{Z}[t]$.*

Proof. By Theorem 6.9, each $\mathfrak{C}_{w_\lambda^{(k)}}$ is an element of the ring generated by ${}_k\vartheta_r^{(l)}$ ($r, l \in \mathbb{Z}$). Since each ${}_k\vartheta_r^{(l)}$ is in the algebra generated by ${}_k\vartheta_r^{(0)}$ ($r \geq 1$) over $\mathbb{Z}[t]$, the claim follows from Proposition 3.7. \square

7. RAISING OPERATORS AND WILSON'S CONJECTURE

7.1. Basics on raising operators. Let $R_{ij}, 1 \leq i < j \leq m$ be the operator that act on \mathbb{Z}^m by

$$R_{ij} : (a_1, \dots, a_i, \dots, a_j, \dots, a_m) \mapsto (a_1, \dots, a_i + 1, \dots, a_j - 1, \dots, a_m).$$

Let $(c^{(1)}, c^{(2)}, \dots, c^{(m)})$ be an m -tuple such that each $c^{(i)}$ is an infinite sequence of variables $c_r^{(i)}$ ($r \in \mathbb{Z}$). The action of R_{ij} on a degree m monomial $c_{r_1}^{(1)} \cdots c_{r_m}^{(m)}$ is defined by

$$R_{ij}(c_{r_1}^{(1)} \cdots c_{r_i}^{(i)} \cdots c_{r_j}^{(j)} \cdots c_{r_m}^{(m)}) = c_{r_1}^{(1)} \cdots c_{r_{i+1}}^{(i)} \cdots c_{r_{j-1}}^{(j)} \cdots c_{r_m}^{(m)}.$$

Let \mathcal{A} be the set of all \mathbb{Z} -linear combinations of monomials $c_{r_1}^{(1)} \cdots c_{r_m}^{(m)}, (r_1, \dots, r_m) \in \mathbb{Z}^m$. The action of any polynomial in R_{ij} on \mathcal{A} is naturally defined. For example,

$$(1 + R_{ij})(c_{r_1}^{(1)} \cdots c_{r_m}^{(m)}) = c_{r_1}^{(1)} \cdots c_{r_i}^{(i)} \cdots c_{r_j}^{(j)} \cdots c_{r_m}^{(m)} + c_{r_1}^{(1)} \cdots c_{r_{i+1}}^{(i)} \cdots c_{r_{j-1}}^{(j)} \cdots c_{r_m}^{(m)}.$$

Since the actions of the operators $R_{ij}, i < j$ commute, they are extended to the action of the polynomial ring $\mathbb{Z}[R_{ij}, 1 \leq i < j \leq m]$. For example, take $(1 + R_{ij})(1 - R_{i'j'}) \in \mathbb{Z}[R_{ij}, 1 \leq i < j \leq m]$

$$\begin{aligned} (1 + R_{ij})(1 - R_{i'j'})(c_{r_1}^{(1)} \cdots c_{r_m}^{(m)}) &= (1 - R_{i'j'})(1 + R_{ij})(c_{r_1}^{(1)} \cdots c_{r_m}^{(m)}) \\ &= (1 - R_{i'j'} + R_{ij} - R_{i'j'}R_{ij})(c_{r_1}^{(1)} \cdots c_{r_m}^{(m)}). \end{aligned}$$

Consider a formal power series $F = \sum_{s=0}^\infty F_s$ where each F_s is a homogeneous polynomial in R_{ij} of degree s , regarding R_{ij} 's as formal variables of degree one. Each F_s acts on \mathcal{A} and so $F_s(c_{r_1}^{(1)} \cdots c_{r_m}^{(m)})$ is in \mathcal{A} . Thus we obtain the following formal series of $c_{r_1}^{(1)} \cdots c_{r_m}^{(m)}, (r_1, \dots, r_m) \in \mathbb{Z}^m$

$$F(c_{r_1}^{(1)} \cdots c_{r_m}^{(m)}) := \sum_{s=0}^\infty F_s(c_{r_1}^{(1)} \cdots c_{r_m}^{(m)}).$$

It is well-defined since the coefficient of each $c_{s_1}^{(1)} \cdots c_{s_m}^{(m)}$ in the sum is finite. Indeed, the degree s of the operator F_s that creates a particular monomial $c_{s_1}^{(1)} \cdots c_{s_m}^{(m)}$ is bounded. It also has the property that the only appearing terms are such that $s_1 + \cdots + s_m = r_1 + \cdots + r_m$. Considering those properties, we can conclude that

$$F(c_{r_1}^{(1)} \cdots c_{r_m}^{(m)}) \Big|_{\geq 0}$$

is a polynomial where $|_{\geq 0}$ denotes the substitution $c_r^{(i)} = 0$ for all $r < 0$ and all i . If F_1 and F_2 are two formal power series as above, then the product $F_1 F_2$ is also such a formal power series and therefore we have

$$(F_1 F_2)(c_{r_1}^{(1)} \cdots c_{r_m}^{(m)}) = F_1(F_2(c_{r_1}^{(1)} \cdots c_{r_m}^{(m)})).$$

The RHS of this identity is well-defined, *i.e.* it is a formal power series such that the coefficient of each $c_{s_1}^{(1)} \cdots c_{s_m}^{(m)}, (s_1, \dots, s_m) \in \mathbb{Z}^m$ is finite.

Example 7.1. The following formal power series is important for our purpose:

$$F := \frac{1 - R_{12}}{1 + R_{12}} = 1 - 2R_{12} + 2R_{12}^2 - \cdots = 1 + \sum_{s=1}^{\infty} (-1)^s 2R_{12}^s.$$

For example, we have $F(c_{-2}^{(1)} c_1^{(2)}) = c_{-2}^{(1)} c_1^{(2)} - 2c_{-1}^{(1)} c_0^{(2)} + 2c_0^{(1)} c_{-1}^{(2)} - 2c_1^{(1)} c_{-2}^{(2)} + \cdots$, and hence we have $F(c_{-2}^{(1)} c_1^{(2)}) \Big|_{\geq 0} = 0$, while $F(c_{-1}^{(1)} c_3^{(2)}) \Big|_{\geq 0} = -2c_0^{(1)} c_2^{(2)} + 2c_1^{(1)} c_1^{(2)} - 2c_2^{(1)} c_0^{(2)}$.

The following lemma is obvious from the definition.

Lemma 7.2. *Let $I(F)$ be the set of i 's such that R_{ij} or R_{ji} appear in F . If $I(F_1) \cap I(F_2) = \emptyset$, then we have the following well-defined identity of formal power series*

$$(F_1 F_2)(c_{r_1}^{(1)} \cdots c_{r_m}^{(m)}) = \left(\prod_{i \notin I(F_1) \cup I(F_2)} c_{r_i}^{(i)} \right) \cdot F_1 \left(\prod_{i \in I(F_1)} c_{r_i}^{(i)} \right) \cdot F_2 \left(\prod_{i \in I(F_2)} c_{r_i}^{(i)} \right).$$

7.2. Pfaffians in terms of raising operators. We have the following description of Pfaffians. Let $\Delta_m := \{(i, j) \in \mathbb{Z}^2 \mid 1 \leq i < j \leq m\}$.

Proposition 7.3. *We have*

$$\text{Pf}[c_{r_1}^{(1)} \cdots c_{r_m}^{(m)}] \Big|_{\geq 0} = \left(\prod_{(i,j) \in \Delta_m} \frac{1 - R_{ij}}{1 + R_{ij}} c_{r_1}^{(1)} \cdots c_{r_m}^{(m)} \right) \Big|_{\geq 0}.$$

Proof. We proceed by induction on m . The cases $m = 1$ is obvious. For $m = 2$, the identity

$$\text{Pf}[c_{r_1}^{(1)} c_{r_2}^{(2)}] \Big|_{\geq 0} = \left(\frac{1 - R_{12}}{1 + R_{12}} (c_{r_1}^{(1)} c_{r_2}^{(2)}) \right) \Big|_{\geq 0}$$

follows clearly from the definition (*cf.* Example 7.1). The general case can be deduced from the following identity of formal series: for m even,

$$\prod_{(i,j) \in \Delta_m} \frac{1 - R_{ij}}{1 + R_{ij}} = \sum_{s=2}^m (-1)^s \frac{1 - R_{1s}}{1 + R_{1s}} \prod_{\substack{(i,j) \in \Delta_m \\ i,j \in \{1, \dots, m\} \setminus \{1, s\}}} \frac{1 - R_{ij}}{1 + R_{ij}},$$

and, for m odd,

$$\prod_{(i,j) \in \Delta_m} \frac{1 - R_{ij}}{1 + R_{ij}} = \sum_{s=1}^m (-1)^{s-1} \prod_{\substack{(i,j) \in \Delta_m \\ i,j \in \{1, \dots, m\} \setminus \{s\}}} \frac{1 - R_{ij}}{1 + R_{ij}}.$$

Since $R_{ij}R_{js} = R_{is}$, the proof of these equations can be reduced to showing the equations for the rational functions obtained from replacing R_{ij} with y_i/y_j . Such equations goes back to Schur ([22, p.226]). \square

7.3. Pfaffian sum formula and Wilson's conjecture.

Definition 7.4. Let λ be a k -strict partition contained in the $(n-k) \times (n+k)$ rectangle and $\chi \in \mathbb{Z}^{n-k}$ the corresponding characteristic index. Let $(\cdot)|_{c=\vartheta(\chi)}$ be the substitution of $\vartheta_{r_i}^{(\chi_i)}$ to $c_{r_i}^{(i)}$ for each $r_i \in \mathbb{Z}$ and $i = 1, \dots, n-k$. Define

$$R_\lambda[\vartheta_{\lambda_1}^{(\chi_1)} \dots \vartheta_{\lambda_{n-k}}^{(\chi_{n-k})}] := \left(\prod_{(i,j) \in D(\lambda)^c} \frac{1 - R_{ij}}{1 + R_{ij}} \prod_{(i,j) \in D(\lambda)} (1 - R_{ij})(c_{\lambda_1}^{(1)} \dots c_{\lambda_{n-k}}^{(n-k)}) \right) \Big|_{c=\vartheta(\chi)}.$$

By Lemma 3.6 and the remark below, this function coincides with the one defined in Wilson's thesis [26, Definition 10]. Note that if $D(\lambda) = \Delta_{n-k}$, it is a single determinant. In fact, the argument in [23, §1] shows

$$\left(\prod_{(i,j) \in \Delta_{n-k}} (1 - R_{ij})(c_{\lambda_1}^{(1)} \dots c_{\lambda_{n-k}}^{(n-k)}) \right) \Big|_{c=\vartheta(\chi)} = \text{Det}[\vartheta_{\lambda_1}^{(\chi_1)} \dots \vartheta_{\lambda_{n-k}}^{(\chi_{n-k})}].$$

If $D(\lambda) = \emptyset$, then the definition gives a single Pfaffian $\text{Pf}[\vartheta_{\lambda_1}^{(\chi_1)} \dots \vartheta_{\lambda_{n-k}}^{(\chi_{n-k})}]$ by Proposition 7.3.

Remark 7.5. Let $\theta_p^r[j]$ be the function defined at Definition 7 in [26], then

$$\theta_p^r[j] = \begin{cases} k\vartheta_{p+j}^r & \text{if } k < p \text{ and } r \leq p - k - 1, \\ k\vartheta_{p+j}^{-r} & \text{if } k \geq p. \end{cases}$$

Finally the following proposition shows that Theorem 1.3 is equivalent to Conjecture 1 in [26], and therefore Corollary 1.5 follows.

Proposition 7.6. Let λ be a k -strict partition in $\mathcal{P}_n^{(k)}$ and χ the corresponding characteristic index. We have

$$\sum_{I \subset D(\lambda)} \text{Pf} \left[\vartheta_{\lambda_1 + a_1^I}^{(\chi_1)} \dots \vartheta_{\lambda_{n-k} + a_{n-k}^I}^{(\chi_{n-k})} \right] = R_\lambda[\vartheta_{\lambda_1}^{(\chi_1)} \dots \vartheta_{\lambda_{n-k}}^{(\chi_{n-k})}].$$

Proof. We have

$$\begin{aligned}
& \sum_{I \subset D(\lambda)} \text{Pf} \left[\vartheta_{\lambda_1 + a_1^I}^{(\chi_1)} \cdots \vartheta_{\lambda_{n-k} + a_{n-k}^I}^{(\chi_{n-k})} \right] \\
&= \left(\prod_{(i,j) \in \Delta_{n-k}} \frac{1 - R_{ij}}{1 + R_{ij}} \sum_{I \subset D(\lambda)} c_{\lambda_1 + a_1^I}^{(1)} \cdots c_{\lambda_{n-k} + a_{n-k}^I}^{(n-k)} \right) \Big|_{c=\vartheta(\chi)} \\
&= \left(\prod_{(i,j) \in \Delta_{n-k}} \frac{1 - R_{ij}}{1 + R_{ij}} \cdot \prod_{(i,j) \in D(\lambda)} (1 + R_{ij}) (c_{\lambda_1}^{(1)} \cdots c_{\lambda_{n-k}}^{(n-k)}) \right) \Big|_{c=\vartheta(\chi)} \\
&= \left(\prod_{(i,j) \in D(\lambda)^c} \frac{1 - R_{ij}}{1 + R_{ij}} \cdot \prod_{(i,j) \in D(\lambda)} (1 - R_{ij}) (c_{\lambda_1}^{(1)} \cdots c_{\lambda_{n-k}}^{(n-k)}) \right) \Big|_{c=\vartheta(\chi)} \\
&= R_\lambda [\vartheta_{\lambda_1}^{(\chi_1)} \cdots \vartheta_{\lambda_{n-k}}^{(\chi_{n-k})}],
\end{aligned}$$

where the first equality follows from the linearity of $(\cdot)|_{c=\vartheta(\chi)}$ and the operators, the second follows from the definition of a^I , the third follows from Lemma 7.2, and the last is the definition of R_λ . \square

8. PFAFFIAN SUM FORMULA BEYOND GRASSMANNIANS

In this section, we show that our technique of deriving the Pfaffian sum formula can be applied beyond the k -Grassmannian elements. Let $G = Sp_{2n}(\mathbb{C})$ as before. First we derive a single Pfaffian formula for the polynomial corresponding to the top class of a symplectic partial flag variety. Then we introduce a certain group of signed permutations, called *pseudo k -Grassmannian elements*, and show the Pfaffian sum formula for each polynomial corresponding to those. We conclude by remarking the possibility of extending our computation further with the example of *all* signed permutations when $n = 3$.

8.1. The longest elements for partial flag varieties. For $J \subset \{0, \dots, n-1\}$, let W_J be the subgroup of W_∞ generated by $s_i, i \notin J$. The set of the minimum-length coset representatives of W_∞/W_J is

$$W^J := \{s \in W_\infty \mid \ell(w) > \ell(ws_i) \ \forall i \geq 0, i \notin J\}.$$

The Schubert varieties of the generalized flag variety G/P_J are indexed by $W_n^J := W_n \cap W^J$ where $G = Sp_{2n}(\mathbb{C})$ as before and P_J is the parabolic subgroup associated to J .

Lemma 8.1. *Let $J = \{k_1 < \dots < k_p\}$. The elements of W_n^J are the signed permutations $w = (w_1 \cdots w_n)$ such that*

$$\begin{cases} w_1 < \dots < w_{k_2}, w_{k_2+1} < \dots < w_{k_3}, \dots, w_{k_p+1} < \dots < w_n & \text{if } k_1 = 0, \\ 0 < w_1 < \dots < w_{k_1}, w_{k_1+1} < \dots < w_{k_2}, \dots, w_{k_p+1} < \dots < w_n & \text{if } k_1 \neq 0. \end{cases} \quad (8.1)$$

Furthermore, the longest element w in W_n^J is given by

$$\begin{cases} w = \overline{k_2} \overline{k_2 - 1} \dots \overline{1} \mid \overline{k_3} \overline{k_3 - 1} \dots \overline{k_2 + 1} \mid \dots \mid \overline{n} \overline{n - 1} \dots \overline{k_p + 1} & \text{if } k_1 = 0, \\ w = 12 \cdots k_1 \mid \overline{k_2} \overline{k_2 - 1} \dots \overline{k_1 + 1} \mid \overline{k_3} \overline{k_3 - 1} \dots \overline{k_2 + 1} \mid \dots \mid \overline{n} \overline{n - 1} \dots \overline{k_p + 1} & \text{if } k_1 \neq 0. \end{cases}$$

Here the vertical lines in one line notation are just only to emphasize the descents. For example, the longest element of $W_7^{\{0,3,5\}}$ is $w = \bar{3}\bar{2}\bar{1}|\bar{5}\bar{4}|\bar{7}\bar{6}$, and the longest element of $W_9^{\{2,6,8\}}$ is $w = 12|\bar{6}\bar{5}\bar{4}\bar{3}|\bar{8}\bar{7}|\bar{9}$.

Proof. Since $W_{n,J} := W_J \cap W_n$ is generated by $s_i, i \in J$, it is obvious that those permutations are coset representatives for both cases $0 \in J$ and $0 \notin J$. Recall that the length of a signed permutation is the “number of inversions” (Proposition 8.1.1 [3])

$$\text{inv}(w) := \text{inv}(w_1, \dots, w_n) + \text{neg}(w_1, \dots, w_n) + \text{nsp}(w_1, \dots, w_n),$$

where neg is the number of negative numbers and nsp is the number of pairs whose sums are negative. Let w be the one at (8.1) and let v be an arbitrary element in the coset wW_J^n . The first part of the claim follows, just by observing that each term in $\text{inv}(v)$ is greater than or equal to the corresponding term of $\text{inv}(w)$. For the second part, it is clear that such w realizes the largest number of inversions, which can be actually computed from the definition of inversions:

$$\text{inv}(w) = \begin{cases} \sum_{i=2}^p k_i(n - k_i) + n + \frac{n(n+1)}{2} & \text{if } k_1 = 0, \\ \sum_{i=1}^p k_i(n - k_i) + n - k_1 + \frac{(n-k_1)(n-k_1+1)}{2} + \frac{k_1(n-k_1)}{2} & \text{if } k_1 \neq 0. \end{cases} \quad (8.2)$$

□

Corollary 8.2. *Let w_0 be the longest element in W_n . Let $J = \{0 = k_1 < \dots < k_p\}$. Let $k_{p+1} = n$. For each $i = 1, \dots, p$, let*

$$\underline{v}_i := (k_{i+1}-1, k_{i+1}-2, \dots, k_i+2, k_i+1, k_{i+1}-1, k_{i+1}-2, \dots, k_i+2, k_{i+1}-1, \dots, k_{i+1}-1, k_{i+1}-2, k_{i+1}-1),$$

where we set $\underline{v}_i = \emptyset$ if $k_{i+1} = k_i + 1$.

- (a) *Let w be the longest element in W_n^J . Let (i_1, \dots, i_r) be a reduced word of w , i.e. $r := \ell(w)$ and $w = s_{i_1} \dots s_{i_r}$. Then*

$$(\underline{v}_1, \dots, \underline{v}_p, i_1, \dots, i_r)$$

is a reduced word for w_0 .

- (b) *Let w' be the longest element in $W_n^{J \setminus \{k_1\}}$ and (i'_1, \dots, i'_r) a reduced word of w' . Let v be the longest element of $W_{k_2}^{(0)}$ and (j_1, \dots, j_s) a reduced word for v . Then*

$$(\underline{v}_1, \dots, \underline{v}_p, j_1, \dots, j_s, i'_1, \dots, i'_r)$$

is a reduced word for w_0 .

Example 8.3. Let $J = \{0\}$ and $n = 4$. We have $\underline{v}_1 = (3, 2, 1, 3, 2, 3)$, $w_0 = s_3 s_2 s_1 s_3 s_2 s_3 w$ and $\ell(w_0) = 6 + \ell(w)$.

Let $J = \{0, 3, 6, 7\}$ and $n = 9$. We have $\underline{v}_1 = (2, 1, 2)$, $\underline{v}_2 = (5, 4, 5)$, $\underline{v}_3 = \emptyset$, and $\underline{v}_4 = (8)$. Thus $w_0 = (s_2 s_1 s_2)(s_5 s_4 s_5) s_8 w$ and $\ell(w_0) = \ell(w) + 7$.

Let $J = \{3, 6, 7\}$ and $n = 9$. Then $w_0 = (s_2 s_1 s_2)(s_5 s_4 s_5) s_8 (s_0 s_1 s_2 s_0 s_1 s_0) w'$ and $\ell(w_0) = \ell(w') + 13$, where w' is the longest element of $W_9^{\{3,6,7\}}$ and $(0, 1, 2, 0, 1, 0)$ is a reduced word of the longest element of $W_3^{(0)}$.

8.2. Pfaffian formula for the longest elements for partial flag varieties. The key fact in this section is that the top function \mathfrak{C}_{w_0} is written in a Pfaffian form. Then we can operate δ_i 's on \mathfrak{C}_{w_0} while keeping the Pfaffian form to some extent. Recall from Remark 4.2 that, for a strict partition $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathcal{P}_\infty^{(0)}$, we have

$$Q_\lambda(x|t) = \text{Pf}[{}_0\vartheta_{\lambda_1}^{\lambda_1-1} {}_0\vartheta_{\lambda_2}^{\lambda_2-1} \cdots {}_0\vartheta_{\lambda_n}^{\lambda_n-1}].$$

Let $\rho = (2n-1, 2n-3, \dots, 3, 1)$. Ikeda-Mihalcea-Naruse [11, Theorem 1.2] proved that for the longest element w_0 in W_n we have

$$\mathfrak{C}_{w_0} = Q_\rho(x|t) \Big|_{(t_1, t_2, t_3, t_4, \dots) = (t_1, -z_1, t_2, -z_2, \dots)}.$$

By observing $Q_\rho(x|t) = \text{Pf}[{}_0\vartheta_{2n-1}^{2n-2} {}_0\vartheta_{2n-3}^{2n-4} \cdots {}_0\vartheta_3^2 {}_0\vartheta_1^0]$ and

$${}_0\vartheta_r^{2i}(x, z|t) \Big|_{(t_1, t_2, t_3, t_4, \dots) = (t_1, -z_1, t_2, -z_2, \dots)} = i\vartheta_r^i(x, z|t),$$

we have the following result.

Theorem 8.4 (Ikeda-Mihalcea-Naruse). *Let $w_0 \in W_n$ be the longest element. Then*

$$\mathfrak{C}_{w_0} = \text{Pf} \left[{}_{n-1}\vartheta_{2n-1}^{(n-1)} {}_{n-2}\vartheta_{2n-3}^{(n-2)} \cdots {}_1\vartheta_3^{(1)} {}_0\vartheta_1^{(0)} \right].$$

The ϑ -functions in the Pfaffian above can be regarded as equivariantly modified special Schubert classes from various isotropic Grassmannians.

To apply δ_i 's systematically to the top function \mathfrak{C}_{w_0} , we need Proposition 5.13, Proposition 5.15, and the following lemma.

Lemma 8.5. *If $l \geq 0$ and $k > 0$,*

$${}_k\vartheta_r^{(l)} = {}_{k-1}\vartheta_r^{(l+1)} + (t_{l+1} + z_k) \cdot {}_{k-1}\vartheta_{r-1}^{(l)}.$$

Moreover, if $l_p \geq 0$ and $k_p > 0$, then

$$\begin{aligned} & \text{Pf}[{}_k\vartheta_{r_1}^{(l_1)} \cdots {}_{k_p}\vartheta_{r_p}^{(l_p)} \cdots {}_{k_m}\vartheta_{r_m}^{(l_m)}] \\ &= \text{Pf}[{}_k\vartheta_{r_1}^{(l_1)} \cdots {}_{k_p-1}\vartheta_{r_p}^{(l_p+1)} \cdots {}_{k_m}\vartheta_{r_m}^{(l_m)}] + (t_{l_p+1} + z_{k_p}) \text{Pf}[{}_k\vartheta_{r_1}^{(l_1)} \cdots {}_{k_p-1}\vartheta_{r_p-1}^{(l_p)} \cdots {}_{k_m}\vartheta_{r_m}^{(l_m)}]. \end{aligned}$$

Proof. The equation ${}_kf_{l+1}(u) = {}_kf_l(u) \cdot (1 - t_{l+1}u) = {}_{k-1}f_{l+1}(u) \cdot (1 + z_ku)$ implies

$${}_k\vartheta_r^{(l+1)} = {}_k\vartheta_r^{(l)} - t_{l+1} \cdot {}_k\vartheta_{r-1}^{(l)} = {}_{k-1}\vartheta_r^{(l+1)} + z_k \cdot {}_{k-1}\vartheta_{r-1}^{(l)}.$$

The first claim follows from using this formula twice. Indeed, we have

$${}_k\vartheta_r^{(l)} - t_{l+1} \cdot ({}_{k-1}\vartheta_{r-1}^{(l)} + z_k \cdot {}_{k-1}\vartheta_{r-2}^{(l)}) = {}_{k-1}\vartheta_r^{(l+1)} + z_k \cdot ({}_{k-1}\vartheta_{r-1}^{(l)} - t_{l+1} \cdot {}_{k-1}\vartheta_{r-2}^{(l)}).$$

This gives the desired equality. The second claim follows from the multilinearity of Pfaffian. \square

We obtain the next lemma by the consecutive application of Proposition 5.15 (b), Lemma 8.5, and Proposition 5.13 in this order.

Lemma 8.6. *Suppose $l_{p+1} + 1 = l_p = i > 0$ and $l_q \notin \{\pm i\}$ for all $q \neq p$. If $(k_{p+1}, r_{p+1}) = (k_p - 1, r_p - 2)$, and $r_p > l_p + k_p$, then*

$$\delta_i \text{Pf}[{}_k\vartheta_{r_1}^{(l_1)} \cdots {}_{k_p}\vartheta_{r_p}^{(i)} {}_{k_{p+1}}\vartheta_{r_{p+1}}^{(i-1)} \cdots {}_{k_m}\vartheta_{r_m}^{(l_m)}] = \text{Pf}[{}_k\vartheta_{r_1}^{(l_1)} \cdots {}_{k_p-1}\vartheta_{r_p-1}^{(i)} {}_{k_{p+1}}\vartheta_{r_{p+1}}^{(i-1)} \cdots {}_{k_m}\vartheta_{r_m}^{(l_m)}].$$

Proof. We have

$$\begin{aligned}
& \delta_i \text{Pf}[k_1 \vartheta_{r_1}^{(l_1)} \cdots k_p \vartheta_{r_p}^{(i)} k_{p+1} \vartheta_{r_{p+1}}^{(i-1)} \cdots k_m \vartheta_{r_m}^{(l_m)}] \\
&= \text{Pf}[k_1 \vartheta_{r_1}^{(l_1)} \cdots k_{p-1} \vartheta_{r_{p-1}}^{(i)} k_{p+1} \vartheta_{r_{p+1}}^{(i-1)} \cdots k_m \vartheta_{r_m}^{(l_m)}] + (t_{i+1} + z_{k_p}) \text{Pf}[k_1 \vartheta_{r_1}^{(l_1)} \cdots k_{p-1} \vartheta_{r_{p-2}}^{(i-1)} k_{p+1} \vartheta_{r_{p+1}}^{(i-1)} \cdots k_m \vartheta_{r_m}^{(l_m)}] \\
&= \text{Pf}[k_1 \vartheta_{r_1}^{(l_1)} \cdots k_{p-1} \vartheta_{r_{p-1}}^{(i)} k_{p+1} \vartheta_{r_{p+1}}^{(i-1)} \cdots k_m \vartheta_{r_m}^{(l_m)}].
\end{aligned}$$

□

Example 8.7. The following computation demonstrates the content of the above lemma.

$$\begin{aligned}
\delta_3 \text{Pf}[3 \vartheta_7^{(3)} 2 \vartheta_5^{(2)}] &= \text{Pf}[3 \vartheta_6^{(2)} 2 \vartheta_5^{(2)}] \\
&= \text{Pf}[2 \vartheta_6^{(3)} 2 \vartheta_5^{(2)}] + (t_3 + z_3) \text{Pf}[2 \vartheta_5^{(2)} 2 \vartheta_5^{(2)}] \\
&= \text{Pf}[2 \vartheta_6^{(3)} 2 \vartheta_5^{(2)}],
\end{aligned}$$

where the first equality follows from Proposition 5.15 (b), the second equality follows from Lemma 8.5, the third is by Proposition 5.13.

Example 8.8. The above lemma allows us to find the Pfaffian formula for the longest element w of $W_n^{(0)}$. By Corollary 8.2 where $J = \{0\}$, we have

$$\mathfrak{C}_w = (\delta_{n-1})(\delta_{n-2} \delta_{n-1}) \cdots (\delta_2 \delta_3 \cdots \delta_{n-2} \delta_{n-1})(\delta_1 \delta_2 \cdots \delta_{n-2} \delta_{n-1}) \mathfrak{C}_{w_0}.$$

Observe that Lemma 8.6 applies to the action of each δ_i , and we have the known formula

$$\mathfrak{C}_w = \text{Pf}[0 \vartheta_n^{(n-1)} 0 \vartheta_{n-1}^{(n-2)} \cdots 0 \vartheta_2^{(1)} 0 \vartheta_1^{(0)}].$$

For example, we compute \mathfrak{C}_{4321} (the longest element in $W_4^{(0)}$) from the top function \mathfrak{C}_{1234} .

$$\begin{aligned}
\mathfrak{C}_{1234} &= \text{Pf}[3 \vartheta_7^{(3)} 2 \vartheta_5^{(2)} 1 \vartheta_3^{(1)} 0 \vartheta_1^{(0)}] \\
\begin{smallmatrix} \delta_3 \\ \rightarrow \end{smallmatrix} \mathfrak{C}_{1243} &= \text{Pf}[2 \vartheta_6^{(3)} 2 \vartheta_5^{(2)} 1 \vartheta_3^{(1)} 0 \vartheta_1^{(0)}] \\
\begin{smallmatrix} \delta_2 \\ \rightarrow \end{smallmatrix} \mathfrak{C}_{1342} &= \text{Pf}[2 \vartheta_6^{(3)} 1 \vartheta_4^{(2)} 1 \vartheta_3^{(1)} 0 \vartheta_1^{(0)}] \\
\begin{smallmatrix} \delta_1 \\ \rightarrow \end{smallmatrix} \mathfrak{C}_{2341} &= \text{Pf}[2 \vartheta_6^{(3)} 1 \vartheta_4^{(2)} 0 \vartheta_2^{(1)} 0 \vartheta_1^{(0)}] \\
\begin{smallmatrix} \delta_3 \\ \rightarrow \end{smallmatrix} \mathfrak{C}_{2431} &= \text{Pf}[1 \vartheta_5^{(3)} 1 \vartheta_4^{(2)} 0 \vartheta_2^{(1)} 0 \vartheta_1^{(0)}] \\
\begin{smallmatrix} \delta_2 \\ \rightarrow \end{smallmatrix} \mathfrak{C}_{3421} &= \text{Pf}[1 \vartheta_5^{(3)} 0 \vartheta_3^{(2)} 0 \vartheta_2^{(1)} 0 \vartheta_1^{(0)}] \\
\begin{smallmatrix} \delta_3 \\ \rightarrow \end{smallmatrix} \mathfrak{C}_{4321} &= \text{Pf}[0 \vartheta_4^{(3)} 0 \vartheta_3^{(2)} 0 \vartheta_2^{(1)} 0 \vartheta_1^{(0)}].
\end{aligned}$$

In general, we can find the Pfaffian formula for the longest element of W_n^J . First we introduce the following notation for simplicity.

Definition 8.9. For each $B = (\kappa_1, \dots, \kappa_b; r_1, \dots, r_b; l_1, \dots, l_b) \in \mathbb{Z}^{3b}$, we formally denote

$$\vartheta_B := \kappa_b \vartheta_{r_b}^{(l_b)} \cdots \kappa_1 \vartheta_{r_1}^{(l_1)}.$$

By choosing any integer sequence $0 \leq k_1 < k_2 < \cdots < k_p < n =: k_{p+1}$, we can write

$$\mathfrak{C}_{w_0} = \text{Pf}[\vartheta_{B_p} \cdots \vartheta_{B_1}],$$

where, for each $i = 1, \dots, p$,

$$B_i := (k_i, k_i + 1, \dots, k_{i+1} - 1; 2k_i + 1, 2k_i + 3, \dots, 2k_{i+1} - 1; k_i, k_i + 1, \dots, k_{i+1} - 1) \in \mathbb{Z}^{3(k_{i+1} - k_i)}.$$

Theorem 8.10. *Let w be the longest element in W_n^J where $J = \{k_1 < \dots < k_p\}$. Let $k_{p+1} := n$. Then*

$$\mathfrak{C}_w = \text{Pf}[\vartheta_{\tilde{B}_p} \cdots \vartheta_{\tilde{B}_2} \vartheta_{\tilde{B}_1}], \quad (8.3)$$

where for each $i = 1, \dots, p$,

$$\tilde{B}_i := (k_i, \dots, k_i ; 2k_i + 1, 2k_i + 2, \dots, k_i + k_{i+1} ; k_i, k_i + 1, \dots, k_{i+1} - 1).$$

Proof. Suppose $k_1 = 0$. By Corollary 8.2, we have

$$\mathfrak{C}_w = \delta_{v_1^c} \cdots \delta_{v_p^c} \mathfrak{C}_{w_0},$$

where

$$\delta_{v_i^c} = \delta_{k_{i+1}-1}(\delta_{k_{i+1}-2} \delta_{k_{i+1}-1}) \cdots (\delta_{k_i+2} \cdots \delta_{k_{i+1}-2} \delta_{k_{i+1}-1})(\delta_{k_i+1} \delta_{k_i+2} \cdots \delta_{k_{i+1}-2} \delta_{k_{i+1}-1}).$$

Lemma 8.6 applies to the action of each δ_i above, and we obtain the claim.

Instead of considering the case when $k_1 \neq 0$, we can assume $J = \{k_2 < \dots < k_p\}$ where $k_2 \neq 0$ and w' is the longest element in W_n^J . We need to show that $\mathfrak{C}_{w'} = \text{Pf}[\vartheta_{\tilde{B}_p} \cdots \vartheta_{\tilde{B}_2}]$. By Corollary 8.2,

$$\mathfrak{C}_{w'} = \delta_{j_s} \cdots \delta_{j_1} \mathfrak{C}_w, \quad (8.4)$$

where (j_1, \dots, j_s) is a reduced word for the longest element of $W_{k_2}^{(0)}$ and w is the longest element in $W_n^{J \cup \{k_1=0\}}$. From the previous case, we have

$$\mathfrak{C}_w = \text{Pf}[\vartheta_{\tilde{B}_p} \cdots \vartheta_{\tilde{B}_2} \vartheta_{\tilde{B}_1}].$$

The divided difference operators in (8.4) act only on the part $\vartheta_{\tilde{B}_1}$ through Proposition 5.15. Together with Remark 5.12, we obtain $\mathfrak{C}_{w'} = \text{Pf}[\vartheta_{\tilde{B}_p} \cdots \vartheta_{\tilde{B}_2}]$. \square

Example 8.11. In the above proof, we operate δ_i 's as in Example 8.8 to each ϑ_{B_i} of the top function \mathfrak{C}_{w_0} to obtain the Pfaffian formula for the longest element of W_n^J for $0 \in J$. For example, we compute $\mathfrak{C}_{\bar{3}2\bar{1}|\bar{6}5\bar{4}|\bar{7}8\bar{9}}$ (the longest element in $W_9^{\{0,3,6,7\}}$) from the top function $\mathfrak{C}_{\bar{1}2\bar{3}4|\bar{5}6\bar{7}8\bar{9}}$.

$$\begin{aligned} \mathfrak{C}_{\bar{1}2\bar{3}4|\bar{5}6\bar{7}8\bar{9}} &= \text{Pf}[8\vartheta_{17}^{(8)} 7\vartheta_{15}^{(7)} 6\vartheta_{13}^{(6)} 5\vartheta_{11}^{(5)} 4\vartheta_9^{(4)} 3\vartheta_7^{(3)} 2\vartheta_5^{(2)} 1\vartheta_3^{(1)} 0\vartheta_1^{(0)}] \\ \xrightarrow{\delta_2 \delta_1 \delta_2} \mathfrak{C}_{\bar{3}2\bar{1}|\bar{4}5\bar{6}78\bar{9}} &= \text{Pf}[8\vartheta_{17}^{(8)} 7\vartheta_{15}^{(7)} 6\vartheta_{13}^{(6)} 5\vartheta_{11}^{(5)} 4\vartheta_9^{(4)} 3\vartheta_7^{(3)} | 0\vartheta_3^{(2)} 0\vartheta_2^{(1)} 0\vartheta_1^{(0)}] \\ \xrightarrow{\delta_5 \delta_4 \delta_5} \mathfrak{C}_{\bar{3}2\bar{1}|\bar{6}5\bar{4}|\bar{7}8\bar{9}} &= \text{Pf}[8\vartheta_{17}^{(8)} 7\vartheta_{15}^{(7)} 6\vartheta_{13}^{(6)} | 3\vartheta_9^{(5)} 3\vartheta_8^{(4)} 3\vartheta_7^{(3)} | 0\vartheta_3^{(2)} 0\vartheta_2^{(1)} 0\vartheta_1^{(0)}] \\ \xrightarrow{\delta_8} \mathfrak{C}_{\bar{3}2\bar{1}|\bar{6}5\bar{4}|\bar{7}|\bar{9}8} &= \text{Pf}[7\vartheta_{16}^{(8)} 7\vartheta_{15}^{(7)} | 6\vartheta_{13}^{(6)} | 3\vartheta_9^{(5)} 3\vartheta_8^{(4)} 3\vartheta_7^{(3)} | 0\vartheta_3^{(2)} 0\vartheta_2^{(1)} 0\vartheta_1^{(0)}]. \end{aligned}$$

From this we can obtain the formula for the longest element of $W_n^{J \setminus \{0\}}$, by further applying δ_i 's, using Proposition 5.15 and Remark 5.12. For example, we obtain $\mathfrak{C}_{123|\bar{6}5\bar{4}|\bar{7}|\bar{9}8}$ (the longest element in $W_9^{\{3,6,7\}}$) from $\mathfrak{C}_{\bar{3}2\bar{1}|\bar{6}5\bar{4}|\bar{7}|\bar{9}8}$:

$$\begin{aligned} \mathfrak{C}_{\bar{3}2\bar{1}|\bar{6}5\bar{4}|\bar{7}|\bar{9}8} &= \text{Pf}[7\vartheta_{16}^{(8)} 7\vartheta_{15}^{(7)} | 6\vartheta_{13}^{(6)} | 3\vartheta_9^{(5)} 3\vartheta_8^{(4)} 3\vartheta_7^{(3)} | 0\vartheta_3^{(2)} 0\vartheta_2^{(1)} 0\vartheta_1^{(0)}] \\ \xrightarrow{\delta_0 \delta_1 \delta_0 \delta_2 \delta_1 \delta_0} \mathfrak{C}_{123|\bar{6}5\bar{4}|\bar{7}|\bar{9}8} &= \text{Pf}[7\vartheta_{16}^{(8)} 7\vartheta_{15}^{(7)} | 6\vartheta_{13}^{(6)} | 3\vartheta_9^{(5)} 3\vartheta_8^{(4)} 3\vartheta_7^{(3)} | 0\vartheta_0^{(-1)} 0\vartheta_0^{(-2)} 0\vartheta_0^{(-3)}] \\ &= \text{Pf}[7\vartheta_{16}^{(8)} 7\vartheta_{15}^{(7)} | 6\vartheta_{13}^{(6)} | 3\vartheta_9^{(5)} 3\vartheta_8^{(4)} 3\vartheta_7^{(3)}]. \end{aligned}$$

The last equality follows from Remark 5.12.

8.3. Pseudo k -Grassmannian elements.

Definition 8.12. Let k be a non-negative integer and $J = \{k = k_1 < \dots < k_p\}$. A *pseudo k -Grassmannian element* is a signed permutation w in W_n^J such that its one line notation is of the form

$$w = w_1 w_2 \dots w_{k_2} | \overline{k_3} \overline{k_3 - 1} \dots \overline{k_2 + 1} | \dots | \overline{n} \overline{n - 1} \dots \overline{k_p + 1}.$$

Note that in this case the first k_2 letters w_1, \dots, w_{k_2} form a k -Grassmannian permutation. Also if we replace them by $\overline{k_2} \overline{k_2 - 1} \dots \overline{k + 1}$ ($k = 0$) or $12 \dots k | \overline{k_2} \overline{k_2 - 1} \dots \overline{k + 1}$ ($k \neq 0$), then we obtain the longest element in W_n^J . For example, $(\bar{2}\bar{1}|\bar{4}\bar{3})$ is pseudo 0-Grassmannian in $W_4^{\{0,2\}}$ and $(2|\bar{3}\bar{1}|\bar{4})$ is pseudo 1-Grassmannian in $W_4^{\{1,3\}}$. On the other hand, $(\bar{3}\bar{1}|\bar{4}\bar{2})$ and $(3|\bar{4}\bar{1}|\bar{2})$ are in $W_4^{\{0,2\}}$ and $W_4^{\{1,3\}}$ respectively but they are not pseudo k -Grassmannian.

Now the following theorem follows from the same argument in proof of Theorem 6.2, just by operating on the part $\vartheta_{\tilde{B}_1}$ in Theorem 8.10 (use Proposition 5.15).

Theorem 8.13. Let w be a pseudo k -Grassmannian element in W_n^J where $J = \{k_1 < \dots < k_p\}$. Let $k = k_1$ and $m = k_2$. Let \tilde{B}_i be as in Theorem 8.10. We have

$$\mathfrak{C}_w = \sum_{I \subset D(\lambda)} \text{Pf} \left[\vartheta_{\tilde{B}_p} \dots \vartheta_{\tilde{B}_2} \vartheta_{\lambda_1 + a_1^I}^{(\chi_1)} \dots \vartheta_{\lambda_{m-k} + a_{m-k}^I}^{(\chi_{m-k})} \right], \quad (8.5)$$

where λ is the k -strict partition in $\mathcal{P}_m^{(k)}$ associated to the signed permutation $(w_1 \dots w_m)$ consisting of the first m letters of w , χ is the associated characteristic index, I runs over all subsets of $D(\lambda)$, and $a_s^I = \#\{j \mid (s, j) \in I\} - \#\{i \mid (i, s) \in I\}$.

8.4. All $n = 3$ signed permutations. We can go further from Theorem 8.13. For example, let's look at

$$\mathfrak{C}_{\bar{2}\bar{1}|\bar{4}\bar{3}} = \text{Pf}[{}_2\vartheta_6^{(3)} {}_2\vartheta_5^{(2)} | {}_0\vartheta_2^{(1)} {}_0\vartheta_1^{(0)}] \xrightarrow{\delta_2} \mathfrak{C}_{\bar{3}\bar{1}|\bar{4}\bar{2}} = \text{Pf}[{}_2\vartheta_6^{(3)} {}_2\vartheta_4^{(1)} | {}_0\vartheta_2^{(1)} {}_0\vartheta_1^{(0)}].$$

The upper indices collide at 1, and we don't have a systematic technique to apply δ_1 without breaking the Pfaffian form. However this shows that the Pfaffian sum formula exists beyond the pseudo k -Grassmannian permutations, since $\bar{3}\bar{1}|\bar{4}\bar{2}$ is not a pseudo k -Grassmannian. We can further apply δ_3 and δ_0 to $\mathfrak{C}_{\bar{3}\bar{1}|\bar{4}\bar{2}}$, keeping the Pfaffian form. For example, we can easily compute all 48 signed permutation of W_3 . Among them, there are 16 permutations that are not pseudo k -Grassmannian permutations. It turns out that all of them are written as (sum of) Pfaffians, *except* $\mathfrak{C}_{\bar{3}\bar{2}\bar{1}}$, $\mathfrak{C}_{\bar{3}\bar{2}\bar{1}}$, $\mathfrak{C}_{\bar{2}\bar{3}\bar{1}}$ and $\mathfrak{C}_{\bar{1}\bar{3}\bar{2}}$.

9. EXAMPLE: $(n, k) = (5, 2), (5, 3)$ **Case** $(n, k) = (5, 3)$

$$\begin{aligned}
& \begin{array}{c} 123|45 \\ (00) \text{ Det}[\vartheta_0^{-4}\vartheta_0^{-5}] \end{array} \\
& \begin{array}{c} 124|35 \\ (10) \text{ Det}[\vartheta_1^{-3}\vartheta_0^{-5}] \end{array} \\
& \begin{array}{cc} 134|25 & 125|34 \\ (20) \text{ Det}[\vartheta_2^{-2}\vartheta_0^{-5}] & (11) \text{ Det}[\vartheta_1^{-3}\vartheta_1^{-4}] \end{array} \\
& \begin{array}{cc} 234|15 & 135|24 \\ (30) \text{ Det}[\vartheta_3^{-1}\vartheta_0^{-5}] & (21) \text{ Det}[\vartheta_2^{-2}\vartheta_1^{-4}] \end{array} \\
& \begin{array}{ccc} 234|\bar{1}5 & 235|14 & 145|23 \\ (40) \text{ Det}[\vartheta_4^0\vartheta_0^{-5}] & (31) \text{ Det}[\vartheta_3^{-1}\vartheta_1^{-4}] & (22) \text{ Det}[\vartheta_2^{-2}\vartheta_2^{-3}] \end{array} \\
& \begin{array}{ccc} 134|\bar{2}5 & 235|\bar{1}4 & 245|13 \\ (50) \text{ Det}[\vartheta_5^1\vartheta_0^{-5}] & (41) \text{ Det}[\vartheta_4^0\vartheta_1^{-4}] & (32) \text{ Det}[\vartheta_3^{-1}\vartheta_2^{-3}] \end{array} \\
& \begin{array}{cccc} 124|\bar{3}5 & 135|\bar{2}4 & 245|\bar{1}3 & 345|12 \\ (60) \text{ Det}[\vartheta_6^2\vartheta_0^{-5}] & (51) \text{ Det}[\vartheta_5^1\vartheta_1^{-4}] & (42) \text{ Det}[\vartheta_4^0\vartheta_2^{-3}] & (33) \text{ Det}[\vartheta_3^{-1}\vartheta_3^{-2}] \end{array} \\
& \begin{array}{cccc} 123|\bar{4}5 & 125|\bar{3}4 & 145|\bar{2}3 & 345|\bar{1}2 \\ (70) \text{ Det}[\vartheta_7^3\vartheta_0^{-5}] & (61) \text{ Det}[\vartheta_6^2\vartheta_1^{-4}] & (52) \text{ Det}[\vartheta_5^1\vartheta_2^{-3}] & (43) \text{ Det}[\vartheta_4^0\vartheta_3^{-2}] \end{array} \\
& \begin{array}{cccc} 123|\bar{5}4 & 125|\bar{4}3 & 145|\bar{3}2 & 345|\bar{2}1 \\ (80) \text{ Pf}[\vartheta_8^4\vartheta_0^{-4}] & (71) \text{ Pf}[\vartheta_7^3\vartheta_1^{-3}] & (62) \text{ Pf}[\vartheta_6^2\vartheta_2^{-2}] & (53) \text{ Pf}[\vartheta_5^1\vartheta_3^{-1}] \end{array} \\
& \begin{array}{cccc} 124|\bar{5}3 & 135|\bar{4}2 & 245|\bar{3}1 & 345|\bar{2}\bar{1} \\ (81) \text{ Pf}[\vartheta_8^4\vartheta_1^{-3}] & (72) \text{ Pf}[\vartheta_7^3\vartheta_2^{-2}] & (63) \text{ Pf}[\vartheta_6^2\vartheta_3^{-1}] & (54) \text{ Pf}[\vartheta_5^1\vartheta_4^0] \end{array} \\
& \begin{array}{ccc} 134|\bar{5}2 & 235|\bar{4}1 & 245|\bar{3}\bar{1} \\ (82) \text{ Pf}[\vartheta_8^4\vartheta_2^{-2}] & (73) \text{ Pf}[\vartheta_7^3\vartheta_3^{-1}] & (64) \text{ Pf}[\vartheta_6^2\vartheta_4^0] \end{array} \\
& \begin{array}{ccc} 234|\bar{5}1 & 235|\bar{4}\bar{1} & 145|\bar{3}\bar{2} \\ (83) \text{ Pf}[\vartheta_8^4\vartheta_3^{-1}] & (74) \text{ Pf}[\vartheta_7^3\vartheta_4^0] & (65) \text{ Pf}[\vartheta_6^2\vartheta_5^1] \end{array} \\
& \begin{array}{cc} 234|\bar{5}\bar{1} & 135|\bar{4}\bar{2} \\ (84) \text{ Pf}[\vartheta_8^4\vartheta_4^0] & (75) \text{ Pf}[\vartheta_7^3\vartheta_5^1] \end{array} \\
& \begin{array}{cc} 134|\bar{5}\bar{2} & 125|\bar{4}\bar{3} \\ (85) \text{ Pf}[\vartheta_8^4\vartheta_5^1] & (76) \text{ Pf}[\vartheta_7^3\vartheta_6^2] \end{array} \\
& \begin{array}{c} 124|\bar{5}\bar{3} \\ (86) \text{ Pf}[\vartheta_8^4\vartheta_6^2] \end{array} \\
& \begin{array}{c} 123|\bar{5}\bar{4} \\ (87) \text{ Pf}[\vartheta_8^4\vartheta_7^3] \end{array}
\end{aligned}$$

Case $(n, k) = (5, 2)$.

$ \begin{array}{c} 12 345 \\ (000) \\ \text{Det}[\vartheta_0^{-3}\vartheta_0^{-4}\vartheta_0^{-5}] \end{array} $							
$ \begin{array}{c} 13 245 \\ (100) \\ \text{Det}[\vartheta_1^{-2}\vartheta_0^{-4}\vartheta_0^{-5}] \end{array} $							
$ \begin{array}{c} 23 145 \\ (200) \\ \text{Det}[\vartheta_2^{-1}\vartheta_0^{-4}\vartheta_0^{-5}] \end{array} $		$ \begin{array}{c} 14 235 \\ (110) \\ \text{Det}[\vartheta_1^{-2}\vartheta_1^{-3}\vartheta_0^{-5}] \end{array} $					
$ \begin{array}{c} 23 \bar{1}45 \\ (300) \\ \text{Det}[\vartheta_3^0\vartheta_0^{-4}\vartheta_0^{-5}] \end{array} $		$ \begin{array}{c} 24 135 \\ (210) \\ \text{Pf}[\vartheta_2^{-1}\vartheta_1^{-3}\vartheta_0^{-5}] \end{array} $		$ \begin{array}{c} 15 234 \\ (111) \\ \text{Det}[\vartheta_1^{-2}\vartheta_1^{-3}\vartheta_1^{-4}] \end{array} $			
$ \begin{array}{c} 13 \bar{2}45 \\ (400) \\ \text{Pf}[\vartheta_4^1\vartheta_0^{-4}\vartheta_0^{-5}] \end{array} $		$ \begin{array}{c} 24 \bar{1}35 \\ (310) \\ \text{Det}[\vartheta_3^0\vartheta_1^{-3}\vartheta_0^{-5}] \end{array} $		$ \begin{array}{c} 34 125 \\ (220) \\ \text{Det}[\vartheta_2^{-1}\vartheta_2^{-2}\vartheta_0^{-5}] \end{array} $		$ \begin{array}{c} 25 134 \\ (211) \\ \text{Det}[\vartheta_2^{-1}\vartheta_1^{-3}\vartheta_1^{-4}] \end{array} $	
$ \begin{array}{c} 12 \bar{3}45 \\ (500) \\ \text{Det}[\vartheta_5^2\vartheta_0^{-4}\vartheta_0^{-5}] \end{array} $		$ \begin{array}{c} 14 \bar{2}35 \\ (410) \\ \text{Det}[\vartheta_4^1\vartheta_1^{-2}\vartheta_0^{-5}] \end{array} $		$ \begin{array}{c} 34 \bar{1}25 \\ (320) \\ \text{Det}[\vartheta_3^0\vartheta_2^{-2}\vartheta_0^{-5}] \end{array} $		$ \begin{array}{c} 25 \bar{1}34 \\ (311) \\ \begin{array}{l} \text{Pf}[\vartheta_3^0\vartheta_1^{-3}\vartheta_1^{-4}] \\ +\text{Pf}[\vartheta_4^0\vartheta_0^{-3}\vartheta_1^{-4}] \\ +\text{Det}[\vartheta_4^0\vartheta_1^{-3}\vartheta_0^{-4}] \\ +\text{Det}[\vartheta_5^0\vartheta_0^{-3}\vartheta_0^{-4}] \end{array} \end{array} $	
$ \begin{array}{c} 12 \bar{4}35 \\ (600) \\ \text{Det}[\vartheta_6^3\vartheta_0^{-3}\vartheta_0^{-5}] \end{array} $		$ \begin{array}{c} 14 \bar{3}25 \\ (510) \\ \text{Pf}[\vartheta_5^2\vartheta_1^{-2}\vartheta_0^{-5}] \end{array} $		$ \begin{array}{c} 34 \bar{2}15 \\ (420) \\ \text{Pf}[\vartheta_4^1\vartheta_2^{-1}\vartheta_0^{-5}] \end{array} $		$ \begin{array}{c} 15 \bar{2}34 \\ (411) \\ \begin{array}{l} \text{Pf}[\vartheta_4^1\vartheta_1^{-3}\vartheta_1^{-4}] \\ +\text{Pf}[\vartheta_5^1\vartheta_0^{-3}\vartheta_1^{-4}] \\ +\text{Det}[\vartheta_5^1\vartheta_1^{-3}\vartheta_0^{-4}] \\ +\text{Det}[\vartheta_6^1\vartheta_0^{-3}\vartheta_0^{-4}] \end{array} \end{array} $	
$ \begin{array}{c} 12 \bar{5}34 \\ (700) \\ \text{Det}[\vartheta_7^4\vartheta_0^{-3}\vartheta_0^{-4}] \end{array} $		$ \begin{array}{c} 13 \bar{4}25 \\ (610) \\ \text{Pf}[\vartheta_6^3\vartheta_1^{-2}\vartheta_0^{-5}] \end{array} $		$ \begin{array}{c} 24 \bar{3}15 \\ (520) \\ \text{Pf}[\vartheta_5^2\vartheta_2^{-1}\vartheta_0^{-5}] \end{array} $		$ \begin{array}{c} 34 \bar{2}15 \\ (430) \\ \text{Pf}[\vartheta_4^1\vartheta_3^0\vartheta_0^{-5}] \end{array} $	
				$ \begin{array}{c} 15 \bar{3}24 \\ (511) \\ \begin{array}{l} \text{Pf}[\vartheta_5^2\vartheta_1^{-2}\vartheta_1^{-4}] \\ +\text{Det}[\vartheta_6^2\vartheta_1^{-2}\vartheta_0^{-4}] \end{array} \end{array} $		$ \begin{array}{c} 35 \bar{2}14 \\ (421) \\ \begin{array}{l} \text{Pf}[\vartheta_4^1\vartheta_2^{-1}\vartheta_1^{-4}] \\ +\text{Det}[\vartheta_5^1\vartheta_2^{-1}\vartheta_0^{-4}] \end{array} \end{array} $	
						$ \begin{array}{c} 45 \bar{1}23 \\ (322) \\ \text{Det}[\vartheta_3^0\vartheta_2^{-2}\vartheta_2^{-3}] \end{array} $	
$ \begin{array}{c} 13 \bar{5}24 \\ (710) \\ \text{Pf}[\vartheta_7^4\vartheta_1^{-2}\vartheta_0^{-4}] \end{array} $		$ \begin{array}{c} 23 \bar{4}15 \\ (620) \\ \text{Pf}[\vartheta_6^3\vartheta_2^{-1}\vartheta_0^{-5}] \end{array} $		$ \begin{array}{c} 24 \bar{3}\bar{1}5 \\ (530) \\ \text{Pf}[\vartheta_5^2\vartheta_3^0\vartheta_0^{-5}] \end{array} $		$ \begin{array}{c} 15 \bar{4}23 \\ (611) \\ \begin{array}{l} \text{Pf}[\vartheta_6^3\vartheta_1^{-2}\vartheta_1^{-3}] \\ +\text{Pf}[\vartheta_6^3\vartheta_2^{-2}\vartheta_0^{-3}] \end{array} \end{array} $	
				$ \begin{array}{c} 25 \bar{3}14 \\ (521) \\ \begin{array}{l} \text{Pf}[\vartheta_5^2\vartheta_2^{-1}\vartheta_1^{-4}] \\ +\text{Det}[\vartheta_6^2\vartheta_2^{-1}\vartheta_0^{-4}] \end{array} \end{array} $		$ \begin{array}{c} 35 \bar{2}\bar{1}4 \\ (431) \\ \begin{array}{l} \text{Pf}[\vartheta_4^1\vartheta_3^0\vartheta_1^{-4}] \\ +\text{Det}[\vartheta_5^1\vartheta_3^0\vartheta_0^{-4}] \end{array} \end{array} $	
						$ \begin{array}{c} 45 \bar{2}13 \\ (422) \\ \begin{array}{l} \text{Pf}[\vartheta_4^1\vartheta_2^{-1}\vartheta_2^{-3}] \\ +\text{Pf}[\vartheta_5^1\vartheta_2^{-1}\vartheta_1^{-3}] \\ +\text{Pf}[\vartheta_4^1\vartheta_3^{-1}\vartheta_1^{-3}] \\ +\text{Pf}[\vartheta_5^1\vartheta_3^{-1}\vartheta_0^{-3}] \end{array} \end{array} $	
$ \begin{array}{c} 23 \bar{5}14 \\ (720) \\ \text{Pf}[\vartheta_7^4\vartheta_2^{-1}\vartheta_0^{-4}] \end{array} $		$ \begin{array}{c} 23 \bar{4}\bar{1}5 \\ (630) \\ \text{Pf}[\vartheta_6^3\vartheta_3^0\vartheta_0^{-5}] \end{array} $		$ \begin{array}{c} 14 \bar{3}\bar{2}5 \\ (540) \\ \text{Pf}[\vartheta_5^2\vartheta_4^1\vartheta_0^{-5}] \end{array} $		$ \begin{array}{c} 14 \bar{5}23 \\ (711) \\ \begin{array}{l} \text{Pf}[\vartheta_7^4\vartheta_1^{-2}\vartheta_1^{-3}] \\ +\text{Pf}[\vartheta_7^4\vartheta_2^{-2}\vartheta_0^{-3}] \end{array} \end{array} $	
				$ \begin{array}{c} 25 \bar{4}13 \\ (621) \\ \begin{array}{l} \text{Pf}[\vartheta_6^3\vartheta_2^{-1}\vartheta_1^{-3}] \\ +\text{Pf}[\vartheta_6^3\vartheta_3^{-1}\vartheta_0^{-3}] \end{array} \end{array} $		$ \begin{array}{c} 25 \bar{3}\bar{1}4 \\ (531) \\ \begin{array}{l} \text{Pf}[\vartheta_5^2\vartheta_3^0\vartheta_1^{-4}] \\ +\text{Det}[\vartheta_6^2\vartheta_3^0\vartheta_0^{-4}] \end{array} \end{array} $	
				$ \begin{array}{c} 45 \bar{3}12 \\ (522) \\ \begin{array}{l} \text{Pf}[\vartheta_5^2\vartheta_2^{-1}\vartheta_2^{-2}] \\ +\text{Pf}[\vartheta_5^2\vartheta_3^{-1}\vartheta_1^{-2}] \end{array} \end{array} $		$ \begin{array}{c} 45 \bar{2}\bar{1}3 \\ (432) \\ \begin{array}{l} \text{Pf}[\vartheta_4^1\vartheta_3^0\vartheta_2^{-3}] \\ +\text{Pf}[\vartheta_5^1\vartheta_3^0\vartheta_1^{-3}] \\ +\text{Pf}[\vartheta_4^1\vartheta_4^0\vartheta_1^{-3}] \\ +\text{Pf}[\vartheta_5^1\vartheta_4^0\vartheta_0^{-3}] \end{array} \end{array} $	
$ \begin{array}{c} 23 \bar{5}\bar{1}4 \\ (730) \\ \text{Pf}[\vartheta_7^4\vartheta_3^0\vartheta_0^{-4}] \end{array} $		$ \begin{array}{c} 13 \bar{4}\bar{2}5 \\ (640) \\ \text{Pf}[\vartheta_6^3\vartheta_4^1\vartheta_0^{-5}] \end{array} $		$ \begin{array}{c} 24 \bar{5}13 \\ (721) \\ \begin{array}{l} \text{Pf}[\vartheta_7^4\vartheta_2^{-1}\vartheta_1^{-3}] \\ +\text{Pf}[\vartheta_7^4\vartheta_3^{-1}\vartheta_0^{-3}] \end{array} \end{array} $		$ \begin{array}{c} 25 \bar{4}\bar{1}3 \\ (631) \\ \begin{array}{l} \text{Pf}[\vartheta_6^3\vartheta_3^0\vartheta_1^{-3}] \\ +\text{Pf}[\vartheta_6^3\vartheta_4^0\vartheta_0^{-3}] \end{array} \end{array} $	
				$ \begin{array}{c} 15 \bar{3}\bar{2}4 \\ (541) \\ \begin{array}{l} \text{Pf}[\vartheta_5^2\vartheta_4^1\vartheta_1^{-4}] \\ +\text{Det}[\vartheta_6^2\vartheta_4^1\vartheta_0^{-4}] \end{array} \end{array} $		$ \begin{array}{c} 35 \bar{4}12 \\ (622) \\ \begin{array}{l} \text{Pf}[\vartheta_6^3\vartheta_2^{-1}\vartheta_2^{-2}] \\ +\text{Pf}[\vartheta_6^3\vartheta_3^{-1}\vartheta_1^{-2}] \end{array} \end{array} $	
						$ \begin{array}{c} 45 \bar{3}\bar{1}2 \\ (532) \\ \text{Pf}[\vartheta_5^2\vartheta_3^0\vartheta_2^{-2}] \end{array} $	

13 524	12 435	24 513	15 423	34 512	35 412	45 321
(740)	(650)	(731)	(641)	(722)	(632)	(542)
$\text{Pf}[\vartheta_7^4 \vartheta_4^1 \vartheta_0^{-4}]$	$\text{Pf}[\vartheta_6^3 \vartheta_5^2 \vartheta_0^{-5}]$	$\text{Pf}[\vartheta_7^4 \vartheta_3^0 \vartheta_1^{-3}]$ $+\text{Pf}[\vartheta_7^4 \vartheta_4^0 \vartheta_0^{-3}]$	$\text{Pf}[\vartheta_6^3 \vartheta_4^1 \vartheta_1^{-3}]$ $+\text{Pf}[\vartheta_6^3 \vartheta_5^1 \vartheta_0^{-3}]$	$\text{Pf}[\vartheta_7^4 \vartheta_2^{-1} \vartheta_2^{-2}]$ $+\text{Pf}[\vartheta_7^4 \vartheta_3^{-1} \vartheta_1^{-2}]$	$\text{Pf}[\vartheta_6^3 \vartheta_3^0 \vartheta_2^{-2}]$ $+\text{Pf}[\vartheta_6^3 \vartheta_4^0 \vartheta_1^{-2}]$	$\text{Pf}[\vartheta_5^2 \vartheta_4^1 \vartheta_2^{-1}]$
	12 534	14 523	15 432	34 512	35 421	45 321
	(750)	(741)	(651)	(732)	(642)	(543)
$\text{Pf}[\vartheta_7^4 \vartheta_5^2 \vartheta_0^{-4}]$	$\text{Pf}[\vartheta_7^4 \vartheta_4^1 \vartheta_1^{-3}]$ $+\text{Pf}[\vartheta_7^4 \vartheta_5^1 \vartheta_0^{-3}]$	$\text{Pf}[\vartheta_6^3 \vartheta_5^2 \vartheta_1^{-2}]$	$\text{Pf}[\vartheta_7^4 \vartheta_3^0 \vartheta_2^{-2}]$ $+\text{Pf}[\vartheta_7^4 \vartheta_4^0 \vartheta_1^{-2}]$	$\text{Pf}[\vartheta_6^3 \vartheta_4^1 \vartheta_2^{-1}]$	$\text{Pf}[\vartheta_5^2 \vartheta_4^1 \vartheta_3^0]$	
	12 543	14 532	34 521	25 431	35 421	
	(760)	(751)	(742)	(652)	(643)	
$\text{Pf}[\vartheta_7^4 \vartheta_6^3 \vartheta_0^{-3}]$	$\text{Pf}[\vartheta_7^4 \vartheta_5^2 \vartheta_1^{-2}]$	$\text{Pf}[\vartheta_7^4 \vartheta_4^1 \vartheta_2^{-1}]$	$\text{Pf}[\vartheta_6^3 \vartheta_5^2 \vartheta_2^{-1}]$	$\text{Pf}[\vartheta_6^3 \vartheta_4^1 \vartheta_3^0]$		
		13 542	24 531	34 521	25 431	
		(761)	(752)	(743)	(653)	
	$\text{Pf}[\vartheta_7^4 \vartheta_6^3 \vartheta_1^{-2}]$	$\text{Pf}[\vartheta_7^4 \vartheta_5^2 \vartheta_2^{-1}]$	$\text{Pf}[\vartheta_7^4 \vartheta_4^1 \vartheta_3^0]$	$\text{Pf}[\vartheta_6^3 \vartheta_5^2 \vartheta_3^0]$		
		23 541	24 531	15 432		
		(762)	(753)	(654)		
	$\text{Pf}[\vartheta_7^4 \vartheta_6^3 \vartheta_2^{-1}]$	$\text{Pf}[\vartheta_7^4 \vartheta_5^2 \vartheta_3^0]$	$\text{Pf}[\vartheta_6^3 \vartheta_5^2 \vartheta_4^1]$			
			23 541	14 532		
			(763)	(754)		
		$\text{Pf}[\vartheta_7^4 \vartheta_6^3 \vartheta_3^0]$	$\text{Pf}[\vartheta_7^4 \vartheta_5^2 \vartheta_4^1]$			
				13 542		
				(764)		
				$\text{Pf}[\vartheta_7^4 \vartheta_6^3 \vartheta_4^1]$		
					12 543	
					(765)	
					$\text{Pf}[\vartheta_7^4 \vartheta_6^3 \vartheta_5^2]$	

REFERENCES

- [1] D. Anderson, Introduction to equivariant cohomology in algebraic geometry. In Contributions to algebraic geometry, EMS Ser. Congr. Rep. Eur. Math. Soc., Zürich, 2012, 71–92.
- [2] D. Anderson and W. Fulton, Degeneracy loci, Pfaffians and vexillary permutations in types B,C, and D, available on arXiv:1210.2066.
- [3] A. Björner and F. Brenti, Combinatorics of Coxeter groups, Graduate Texts in Math. **231**, Springer 2005.
- [4] A. S. Buch, A. Kresch, and H. Tamvakis, A Giambelli formula for isotropic Grassmannians, available on arXiv: 0811.2781.
- [5] A. S. Buch, A. Kresch, and H. Tamvakis, A Giambelli formula for even orthogonal Grassmannians, to appear in J. reine angew. Math.
- [6] A. S. Buch, A. Kresch, and H. Tamvakis, Quantum Giambelli formulas for isotropic Grassmannians, Math. Annalen **354** (2012), 801–812.
- [7] W. Fulton and P. Pragacz, *Schubert varieties and degeneracy loci*, Lecture Notes in Math. **1689**, Springer-Verlag, Berlin, 1998.
- [8] G. Z. Giambelli, Risoluzione del problema degli spazi secanti, Mem. R. Accad. Sci. Torino (2) **52** (1902), 171–211.
- [9] J. E. Humphreys, *Reflection Groups and Coxeter Groups* (Cambridge Studies in Advanced Mathematics, No. 29).
- [10] T. Ikeda, Schubert classes in the equivariant cohomology of the Lagrangian Grassmannian, Adv. Math. **215** (2007), 1–23.
- [11] T. Ikeda, L. Mihalcea, and H. Naruse, Double Schubert polynomials for the classical groups, Adv. Math. **226** (2011), 840–886.
- [12] T. Ikeda and H. Naruse, Excited Young diagrams and equivariant Schubert calculus, Trans. Amer. Math. Soc. **361** (2009), 5193–5221.
- [13] V. N. Ivanov, Interpolation analogue of Schur Q -functions, Zap. Nauc. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. **307** (2004), 99–119.
- [14] G. Kempf and D. Laksov, The determinantal formula of Schubert calculus. Acta Math. **132** (1974), 153–162.
- [15] M. Kazarian, On Lagrange and symmetric degeneracy loci, preprint. available at: <http://www.newton.cam.ac.uk/preprints2000.html>.
- [16] A. Kresch and H. Tamvakis, Double Schubert polynomials and degeneracy loci for the classical groups, Annales de l’institut Fourier, **52** no. 6 (2002), 1681–1727.
- [17] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd edition, Oxford Univ. Press, Oxford 1995.
- [18] L. Manivel, *Symmetric Functions, Schubert Polynomials and Degeneracy Loci*, SMF/AMS TEXT and MONO-GRAPHS, vol. **6**, 2001.
- [19] H. Naruse, private communication.
- [20] P. Pragacz, Algebro-geometric applications of Schur S - and Q -polynomials, in *Séminaire d’Algèbre Dubreil-Malliavin 1989-1990*, Springer Lecture Notes in Math. **1478** (1991), 130–191.
- [21] P. Pragacz and J. Ratajski, Formulas for Lagrangian and orthogonal degeneracy loci; \tilde{Q} -polynomial approach, Compositio Math. **107** (1997), no. 1, 11–87.
- [22] I. Schur, Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen, J. reine angew. Math. **139** (1911), 155–250.
- [23] H. Tamvakis, Giambelli and degeneracy locus formulas for classical G/P spaces, available on arXiv: 1305.3543.
- [24] H. Tamvakis, A Giambelli formula for classical G/P spaces. J. Algebraic Geom. **23** (2014), no. 2, 245–278.
- [25] H. Tamvakis, E. Wilson. Double theta polynomials and equivariant Giambelli formulas, available on arXiv:1410.8329.
- [26] E. Wilson, Equivariant Giambelli Formulae for Grassmannians, Ph.D. Thesis, University of Maryland (2010).

DEPARTMENT OF APPLIED MATHEMATICS, OKAYAMA UNIVERSITY OF SCIENCE, OKAYAMA 700-0005, JAPAN

E-mail address: `ike@xmath.ous.ac.jp`

DEPARTMENT OF MATHEMATICAL SCIENCES, KAIST, DAEJEON, SOUTH KOREA

E-mail address: `tomoomatsumura@kaist.ac.kr`